NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

MA3227 Numerical Analysis II

December 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **FIVE (5)** printed pages.
2. Answer **ALL** questions in **Section A**. Each question in Section A carries 15 marks.
3. Answer not more than **TWO (2)** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **ALL** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [15 marks]

(a) Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 2+q & -1+p & & & & \\ -1-p & 2+q & -1+p & & & \\ & -1-p & 2+q & -1+p & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1-p & 2+q & -1+p \\ & & & & -1-p & 2+q \end{pmatrix}$$

is an $N \times N$ tri-diagonal matrix. If $q > 0$ and $|p| \leq 1$, prove that the sequence produced by Gauss-Jacobi method converges to the solution of $Ax = b$ for any starting vector.

(b) Let Q be any $N \times N$ matrix. Prove that if $\rho(Q) < 1$, then $I - Q$ is invertible and

$$(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k,$$

where I is the identical matrix.

Question 2 [15 marks]

Consider the function $f(x) = 2 + x - \tan^{-1}(x)$.

(a) Given any $x, y \in \mathbb{R}$, prove that

$$|f(x) - f(y)| < |x - y|.$$

(b) Determine whether there exists a fixed point $c \in \mathbb{R}$ of f, that is,

$$f(c) = c.$$

Justify your answer.

Question 3 [15 marks]
Consider the well-posed initial value problem:

$$\begin{cases} y'(t) = f(t, y), \quad a \le t \le b, \\ y(a) = \alpha. \end{cases} \tag{1}$$

(a) Consider the following two-step method for solving (1):

$$\begin{cases} \omega_0 = \alpha, \quad \omega_1 = \alpha_1, \\ \omega_{i+1} = \omega_{i-1} + 2hf(t_i, \omega_i), \quad i = 1, 2, \cdots, N-1, \end{cases}$$

where $h = \frac{b-a}{N}$, $t_i = a + ih$ and $\omega_i \approx y(t_i)$ for $i = 0, 1, \cdots, N$. Determine the order of the local truncation error of the given method. Justify your answer.

(b) Prove that the local truncation error of the modified Euler's method is of order 2.

Question 4 [15 marks]
Consider the two-point boundary-value problem:

$$\begin{cases} y''(t) - 37t^2 y'(t) = 95, \quad 6 \le t \le 12, \\ y(6) = 1, \quad y(12) = 2. \end{cases} \tag{2}$$

(a) Solve (2) for $y(9)$ by using the central finite difference method with a step size $h = 3$.

(b) Show how the linear shooting method can be applied to solve (2).

Section B

Answer not more than **TWO** *questions in this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

Consider the well-posed initial value problem:

$$\begin{cases} y'(t) = f(t, y), \quad 0 \le t \le 1, \\ y(0) = y_0. \end{cases}$$

For any given positive integer N, let $h = \frac{1}{N}$ and $t_i = ih$ for $i = 0, 1, \cdots, N$.

(a) Determine the order of the local truncation error of the following Adams-Bashforth method:

$$\begin{cases} \omega_0 = y_0, \quad \omega_1 = y_1, \\ \omega_{i+1} = \omega_i + h(\frac{3}{2} f(t_i, \omega_i) - \frac{1}{2} f(t_{i-1}, \omega_{i-1})), \quad i = 1, 2, \cdots, N-1. \end{cases}$$

(b) Determine the values of a, b and c such that the local truncation error of the following implicit method

$$\begin{cases} \omega_0 = y_0, \quad \omega_1 = y_1, \\ \omega_{i+1} = \omega_i + h[af(t_{i+1}, \omega_{i+1}) + bf(t_i, \omega_i) + cf(t_{i-1}, \omega_{i-1})], \quad i = 1, 2, \cdots, N-1, \end{cases}$$

is of order 3.

Question 6 [20 marks]

Consider solving a non-linear equation $f(x) = 0$, where $f : \mathbb{R} \to \mathbb{R}$ is an infinitely differentiable function on $\mathbb{R}$.

(a) Determine the local convergence rate of the following iteration formula:

$$x_{n+1} = \frac{f(x_n)x_{n-1} - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad n = 1, 2, \cdots.$$

(b) Steffensen's method is defined by the following iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad n = 0, 1, \cdots.$$

where

$$g(x) = (f(x + f(x)) - f(x))/f(x).$$

Show that Steffensen's method is quadratically convergent.

Question 7 [20 marks]
Consider a well-posed initial value problem:

$$\begin{cases} y'(t) = f(t, y), \quad 0 \le t \le 1, \\ y(0) = \alpha. \end{cases} \tag{3}$$

Let $D = \{(t, y), \quad 0 \le t \le 1, \quad -\infty < y < \infty\}$. Suppose that f is continuous on D and satisfies a Lipschitz condition in the variable y on D. Consider the following m-step explicit method for solving (3):

$$\begin{cases} \omega_0 = \alpha, \quad \omega_1 = \alpha_1, \quad \cdots, \quad \omega_{m-1} = \alpha_{m-1}, \\ \omega_{i+1} = a\omega_i + (1-a)\omega_{i-1} + h(\sum_{j=1}^{m} b_{j-1} f(t_{i+1-j}, \omega_{i+1-j})), \quad i = m-1, \cdots, N-1, \end{cases} \tag{4}$$

where $h = \frac{1}{N}$ and $t_i = ih$ for $i = 0, 1, \cdots, N$.

(a) Prove that the method (4) is consistent if and only if

$$a + \sum_{j=1}^{m} b_{j-1} = 2.$$

(b) Prove that the method (4) is stable if and only if

$$0 \le a < 2.$$

END OF PAPER