NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2008-2009

MA2101 Linear Algebra II

November/December 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **ELEVEN (11)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer all questions.

Question 1 [8 marks] For each of the following, determine if it is true or false. There is no need to justify your answer.

(a) For any matrix A, if A is diagonalizable, then all its eigenvalues are distinct.

(b) Suppose V is infinite dimensional and S is any infinite linearly independent set. Then S is a basis of V.

(c) For any subspaces U, W, W' in a vector space V, if $U \oplus W = U \oplus W'$, then $W = W'$.

(d) For any orthogonal matrix A, A^2 is also orthogonal.

Question 2 [8 marks] In $\mathbb{R}^2$, we define

$$(a, b) + (c, d) = (a + c, b + d) \text{ and } (a, b) \times (c, d) = (ac, bd)$$

for all $(a, b), (c, d) \in \mathbb{R}^2$.

(i) Find the multiplicative identity and additive identity in $\mathbb{R}^2$.

(ii) Prove that $(\mathbb{R}^2, +, \times)$ is not a field. Note that A1-A4 are satisfied.

Question 3 [8 marks] Let $\boldsymbol{A} \in \mathcal{M}_{nn}(\mathbb{R})$ and $W = \{\boldsymbol{B} \in \mathcal{M}_{nn}(\mathbb{R}) \mid \boldsymbol{AB} = \boldsymbol{BA}\}$. Suppose $m_{\boldsymbol{A}}(x)$ is the minimal polynomial of A and

$$m_{\boldsymbol{A}}(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_0.$$

(i) Prove that W is a subspace of $\mathcal{M}_{nn}(\mathbb{R})$.

(ii) Prove that $\boldsymbol{I}$, $\boldsymbol{A}$, $\boldsymbol{A}^2$, $\ldots$, $\boldsymbol{A}^{r-1}$ are linearly independent vectors contained in W.

Question 4 [6 marks] Let V be a finite dimensional space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis of V. For any positive integer r with $r < n$, find a subspace of dimension r. Note that you need to justify why the subspace constructed is of dimension r.

Question 5 [10 marks] Let $T : \mathbb{R}^n \to P_m(\mathbb{R})$ be linear transformation. Here $P_m(\mathbb{R})$ denotes the set of real polynomials with degree less than or equal to m.

(i) Find the dimensions of $\mathbb{R}^n$ and $P_m(\mathbb{R})$ over $\mathbb{R}$. [No need to justify your answers.]

(ii) Prove that if $m + 1 < n$, then $Ker(T) \neq \{0\}$.

(iii) Let B_1 and B_2 be bases in $\mathbb{R}^n$ and $P_m(x)$ respectively. Prove that if $m + 1 = n$ and the null space of $[T]_{B_2,B_1} = \{0\}$, then T is bijective.

Question 6 [8 marks] Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the ordered bases for $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$ given by

$$\mathcal{B}_1 = \{1, x, x^2\} \quad \text{and} \quad \mathcal{B}_2 = \{1, x, x^2, x^3\}$$

respectively. Note that $P_2(\mathbb{R})$ and $P_3(\mathbb{R})$ are the set of polynomials with degree less than or equal to 2 and 3 respectively.

(i) Let $T_1 : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be given by

$$T_1(a + bx + cx^2) = (b - a) + (a - b + c)x + cx^2 + (b + 2c)x^3.$$

Find the matrix $[T_1]_{\mathcal{B}_2,\mathcal{B}_1}$.

(ii) Let $T_2 : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation such that

$$[T_2]_{\mathcal{B}_1,\mathcal{B}_2} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Find the matrix $[T_2 \circ T_1]_{\mathcal{B}_1}$.

Question 7 [12 marks] Let V be a vector space and $T : V \to V$ be a linear transformation.

(i) For any positive integers i, j, prove that $KerT^i \subset KerT^j$ if $i < j$.

(ii) Prove that if $\{u_1, u_2, u_3\}$ is a basis of V with

$$T(u_1) = u_2, T(u_2) = u_3 \text{ and } T(u_3) = 0,$$

then $T^3 = 0$ and $KerT \neq KerT^2 \neq KerT^3 = V$.

(iii) Find a basis for $KerT, KerT^2$ and $KerT^3$.

Question 8 [10 marks]

Let A be the real matrix $\begin{pmatrix} 1 & 1 & a \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

(i) Find the characteristic polynomial of A.

(ii) Find all the values of a for which A is diagonalizable.

(iii) For those values of a for which A is not diagonalizable, find all possible Jordan forms.

Question 9 [10 marks] Let $\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 1 & -1 \\ 2 & 2 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

(i) Compute the characteristic polynomial of $\boldsymbol{A}$ and find all its roots. Hence, find all possible minimal polynomial for A.

(ii) Find the minimal polynomial of A and hence find a Jordan canonical form of A.

Question 10 [10 marks] In $\mathbb{R}^4$, we define $\langle (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3 + u_4v_4$ for any $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$.

(i) Prove that $\langle ., . \rangle$ defines an inner product on $\mathbb{R}^4$.

(ii) Let $W = span\{(1, -2, 1, -1), (2, -3, 2, -3), (3, -5, 3, -4), (-1, 1, -1, 2)\}$. Find $W^\perp$ with respect to the inner product defined by $\langle (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3 + u_4v_4$ for any $(u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$.

Question 11 [10 marks] Let A be a symmetric matrix in $M_4(\mathbb{R})$ with eigenvalues λ_1, λ_2. Let E_1 and E_2 be the eigenspaces corresponding to λ_1 and λ_2 respectively.

Suppose $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} \right\}$ is a basis for E_1 and $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for E_2.

(i) Find an orthonormal basis (with respect to the Euclidean inner product) of $\mathbb{R}^4$ consisting of only eigenvectors of A.

(ii) Find an orthogonal matrix that diagonalizes A.

[END OF PAPER]