NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008 - 2009

MA1507 Advanced Calculus

November/December 2008 —— Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **TWO (2)** sections and comprises **FIVE (5)** printed pages. Section A contains **FIVE (5)** questions and Section B contains **THREE (3)** questions.

2. Answer **ALL** questions in Section A, and **TWO (2)** in Section B. The mark for each question is indicated at the beginning of the question.

3. Candidates may use calculators or computers. However, they should lay out systematically the various steps in the calculations to show that they have understood the mathematics.

4. This is an open book examination. Candidates are allowed to refer to any learning material.

• ~ · ~ · ~ SECTION A ~ · ~ · ~ ·

Section A contains **FIVE (5)** questions. Answer **ALL** the questions.

Question 1 (15 marks)

(a) Let $f(x, y, z) = xyz$ and $\boldsymbol{p}_0 = (x_0, y_0, z_0) \neq (0, 0, 0)$.

(i) Find f_x, f_y, f_z.

(ii) Find $\boldsymbol{p}_0 \cdot \nabla f(\boldsymbol{p}_0)$.

(iii) Hence, or otherwise, find the directional derivative of f at the point $\boldsymbol{p}_0$ in the direction of the vector $\boldsymbol{p}_0$.

(iv) Deduce the value of the directional derivative of f at the point $(1, 1, 1)$ in the direction of the vector $(1, 1, 1)$.

(b) If $x^2 + y^2 + z^2 + u^2 - 10 = 0$ and $3x + 3y + z^3 + u^3 = 0$, find
$\left(\frac{\partial z}{\partial x}\right)_y$, $\left(\frac{\partial z}{\partial y}\right)_x$, $\left(\frac{\partial u}{\partial x}\right)_y$, $\left(\frac{\partial u}{\partial y}\right)_x$.

Question 2 (10 marks)

Find the critical points of the function $f(x, y) = xy\, e^{-x-y}$ and determine whether they are local maximum or minimum points or saddle points.

Question 3 (15 marks)

Evaluate

(a) $\iint_D xy \; dxdy$, where D is the region in the first quadrant bounded by the Y-axis, the line $y = 1$ and the curve $y = x^2$.

(b) $\iiint_D z \; dxdydz$, where $D = \{ (x, y, z) : x^2 + y^2 \leq z^2,\ 0 \leq z \leq 1 \}$.

Question 4 (10 marks)

Evaluate $\oint_C \left(x^2 - 2y + x^3 y^4\right) dx + y^3 \left(x^4 + y\right) dy$, where C is the boundary of the square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$.

Question 5 (10 marks)

Evaluate the surface integral, $\iint_S \boldsymbol{F} \cdot \boldsymbol{n} dS$, where $\boldsymbol{F} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ and S is the surface $x^2 + y + z = 1$ in the first octant .

• ~ · ~ · ~ END OF SECTION A ~ · ~ · ~ ·

· ~ · ~ · ~ SECTION B ~ · ~ · ~ ·

Section B contains **THREE (3)** questions. Answer **ANY TWO (2)** questions.

Question 6 (20 marks)

(a) Let S be a smooth surface, A a point not in S and B a point in S nearest to A. State and explain the relationship between the vector $\overrightarrow{AB}$ and the normal to the surface S at B .

(b) Let $\mathscr{P}$ be a plane in $\mathbb{R}^3$, $A = (a, b, c)$ a point not in $\mathscr{P}$ and $B = (1, 1, 2)$ a point in $\mathscr{P}$ nearest to A.

(i) Find the equation of $\mathscr{P}$ in terms of a , b, c .

(ii) If $\mathscr{P}$ is the tangent plane of the paraboloid $z = x^2 + y^2$ at the point B, find the equation of the straight line through B on which A lies. Find all possible values of a, b, c, if the distance of A from B is 2.

(iii) If $\mathscr{P}$ is the tangent plane of the paraboloid $z = x^2 + y^2$ at the point B and A is the point below B, with respect to the XY-plane, and 2 units from B, find the point on the sphere with centre at A and radius 1 that is nearest to the paraboloid.

Question 7 (20 marks)

Let $\boldsymbol{r} = (x, y, z)$ and $\phi(x, y, z) = \rho^n$, where $\rho := \|\boldsymbol{r}\| = \sqrt{x^2 + y^2 + z^2}$ and n is an integer. Note that if n is negative, $\phi(x, y, z)$ is not defined at $(0, 0, 0)$.

(i) Find $\textbf{Curl}(\boldsymbol{r})$ and $\mathrm{div}(\boldsymbol{r})$.

(ii) Find $\boldsymbol{\nabla}\phi$ in terms of ρ and $\boldsymbol{r}$.

(iii) Hence or otherwise, find $\textbf{Curl}(\phi\boldsymbol{r})$ and $\mathrm{div}(\phi\boldsymbol{r})$.

Question 7 (continue)

(iv) Let $\boldsymbol{F} = \dfrac{\boldsymbol{r}}{\rho^3}$, $(x, y, z) \neq 0$. Find

(1) **Curl**$(\boldsymbol{F})$ and $div(\boldsymbol{F})$,

(2) $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r}$ for any piecewise smooth closed curve C that does not pass through the origin,

(3) $\int_C \boldsymbol{F} \cdot d\boldsymbol{r}$, where C consists of straight line segments from the point $(3, 0, 4)$ to $(1, 1, 0)$ to $(0, 1, 2)$,

(4) $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$, where S is a closed surface enclosing the origin and oriented with outward normal $\boldsymbol{n}$.

Question 8 (20 marks)

The plane $x + y + z = 3$ intersects the paraboloid $2z - x^2 - y^2 = 0$ at a curve C.

(a) Find the lowest point L and the highest point H in C from the XY-plane.

(b) Consider the solid bounded by the paraboloid and the plane.

(i) Show that the " roof " of the solid is the section of the plane $z = 3 - x - y$, $(x, y) \in D$, where the domain D is a disc with centre $(-1, -1)$ and radius $2\sqrt{2}$.

(ii) Find the volume of the solid.

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