NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

MA1104 Multivariable Calculus

Nov/Dec 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR(4)** printed pages.

2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **ALL** *questions. Marks for each question are indicated at the beginning of the question.*

Question 1 [15 marks]

(a) Prove that $(a^2+b^2)(c^2+d^2) \geq (ac+bd)^2$ for all real numbers a, b, c, d.

(Hint: Let $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ and $\mathbf{v} = c\mathbf{i} + d\mathbf{j}$.)

(b) Let $\mathbf{u}$ be a unit vector. Prove that there exists a constant C such that

$$\mathbf{u} \times [\mathbf{u} \times (\mathbf{u} \times \mathbf{v})] \cdot \mathbf{w} = C\mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$$

for any vectors $\mathbf{v}$ and $\mathbf{w}$. Find the value of C.

Question 2 [15 marks]

(a) The surfaces

$$f(x,y,z) = x^2 + y^2 - 2 = 0$$

and

$$g(x,y,z) = x + z - 4 = 0$$

meet in an ellipse E. Find the parametric equation of the tangent line to E at the point $(1, 1, 3)$.

(b) Find an equation of the plane containing the lines $x = 2y = 2 - 2z$ and $x = y = 2z - 2$.

Question 3 [15 marks]

(a) The temperature function $T(x,y)$ is defined on $\mathbb{R}^2$ by

$$T(x,y) = x\,\sin 2y,$$

where T is measured in degrees Celsius and x, y are measured in centimetres. An insect is moving clockwise around the circle of radius $1\,cm$ centred at the origin at a speed of $2\,cm/sec$. How fast is the temperature (experienced by the insect) changing in degrees Celsius per second at the point $(1/2, \sqrt{3}/2)$?

(b) Find the point on the graph of $z = x^2 + y^2 + 10$ that is nearest to the plane $x + 2y - z = 0$.

Question 4 [15 marks]

(a) Evaluate the integral $\int_0^3 \int_{\sqrt{x/3}}^1 e^{y^3}\, dydx$.

(b) Let f be the function defined on $\mathbb{R}^2$ by

$$f(x,y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}.$$

(i) Find $f_x(0,0)$ and $f_y(0,0)$.

(ii) Given that

$$f_x(x,y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$$

and

$$f_y(x,y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$$

where $(x,y) \neq (0,0)$. Find $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Is f differentiable at $(0,0)$? Justify your answer.

Question 5 [20 marks]

(a) Let $\mathbf{F}(x,y,z) = \dfrac{-y}{x^2+y^2}\,\mathbf{i} + \dfrac{x}{x^2+y^2}\,\mathbf{j} + z\,\mathbf{k}$.

(i) Find $\operatorname{curl} \mathbf{F}$.

(ii) Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where C is the circle $x^2 + y^2 = 1$ in the xy-plane.

(b) Suppose $f(x,y)$ and its first and second partial derivatives are continuous throughout a disc centred at (a,b). If (a,b) is a critical point of f and $f_{xx}(a,b)$ and $f_{yy}(a,b)$ differ in sign, can you conclude anything about $f(a,b)$? Give reasons for your answer.

(c) Find the region R in the xy-plane such that the value of

$$\iint_R (4 - x^2 - 2y^2)\, dx\, dy$$

is a maximum. Give reasons for your answer.

Question 6 [20 marks]

(a) A region D in space is enclosed by the oriented surface S with outward normal $\mathbf{n}$. Prove that the volume V of D is given by

$$V = \frac{1}{3}\iint_S \mathbf{r} \cdot \mathbf{n}\, dS,$$

where $\mathbf{r}$ is the position vector of the point (x, y, z) in D.

(b) Give an example of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that $\mathbf{F} \neq \mathbf{0}$, $\operatorname{div} \mathbf{F} = 0$ and $\operatorname{curl} \mathbf{F} = \mathbf{0}$.

(c) Evaluate the surface integral $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$$

and S is the oriented surface (with outward normal) of the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2$.

END OF PAPER

Formula List

1. $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} =< a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1 >,$

   where $\mathbf{a} =< a_1, a_2, a_3 >$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$.

2. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

3. The equation of the plane through $P_0 = (x_o, y_0, z_0)$ with normal $< a, b, c >$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

4. Let C be a curve given by $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$. Then

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2}.$$

5. Let $f(x, y)$ be defined in a neighbourhood of (a, b). Then

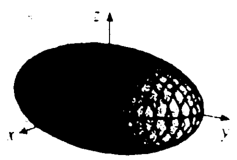
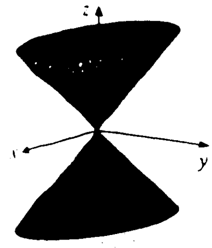
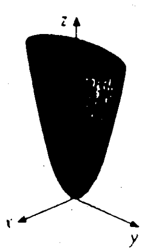
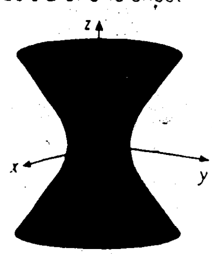
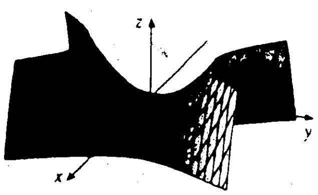
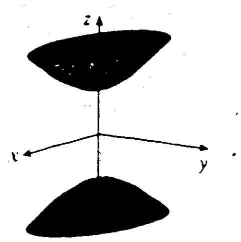
$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if and only if given $\epsilon > 0$, we can find a $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

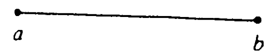
whenever

$$0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta.$$

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

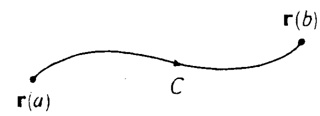
Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$



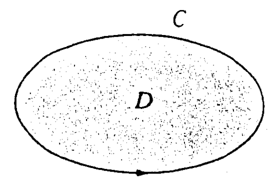
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



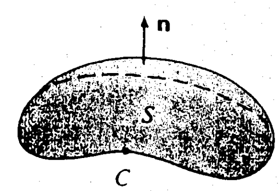
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$