NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION (2008–2009)

MA1102R Calculus

November/December 2008 — Time allowed : 2 hours

INSTRUCTION TO CANDIDATES

1. This examination paper consists of **ONE (1)** section. It contains a total of **EIGHT (8)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for questions are not necessarily the same; marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [18 marks]

Evaluate the following limits.

(a) $\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7}$

(b) $\lim_{x \to \infty} x^{1/x^2}$

(c) $\lim_{x \to 0} \frac{x^2 \sin(1/x)}{\sin x}$

Question 2 [12 marks]

Evaluate the following definite integrals.

(a) $\int_1^4 \frac{(1+\sqrt{x})^4}{\sqrt{x}}\, dx$

(b) $\int_0^1 \left(\sin^{-1}(x)\right)^2 dx$

Question 3 [18 marks]

Determine whether each of the following series is convergent or divergent. Justify your answer.

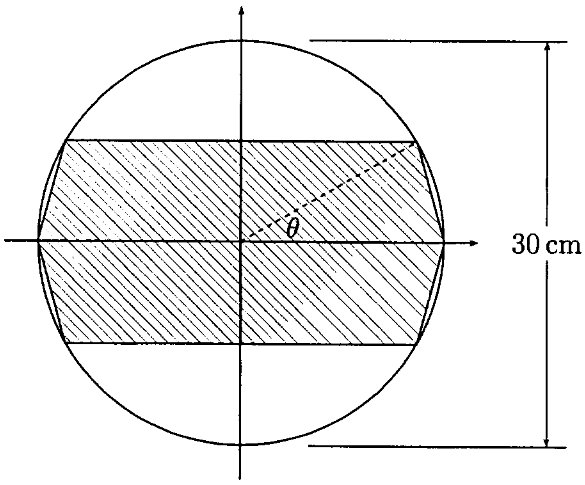
(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n} + \sqrt{n+1}}$

(b) $\sum_{n=3}^{\infty} \frac{1}{(\ln \ln n)^{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}}$

(c) $\sum_{n=1}^{\infty} \left\{ \tan\left(\frac{1}{n^{2/3}}\right) - \tan\left(\frac{1}{n^{2/3}+1}\right) \right\}$

Question 4 [10 marks]

A logger must cut a six-sided beam from a circular log of diameter 30 cm so its cross section is as shown in the following figure. The beam is symmetrical. Show that the area of the cross section is maximal when the cross section is a regular hexagon.



Question 5 [10 marks]

Find the arc length of the curve

$$y = \frac{x^3}{6} + \frac{1}{2x}$$

from $x = 2$ to $x = 3$.

Question 6 [10 marks]

Let

$$u = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}, \quad v = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!}, \quad \text{and} \quad w = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}.$$

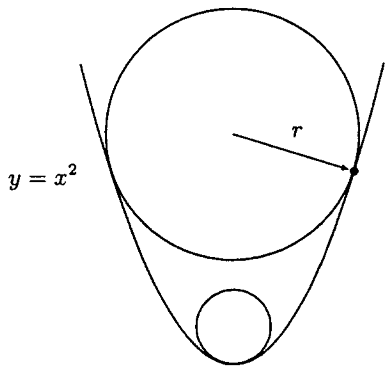
(a) Find the radius of convergence of the power series u.

(b) Show that

$$u^3 + v^3 + w^3 - 3uvw = 1.$$

Question 7 [12 marks]

A circle of radius r is dropped into the parabola $y = x^2$. If r is too large, the circle will not fall all the way to the bottom; if r is sufficiently small, the circle will touch the parabola at its vertex $(0, 0)$. Find the largest value of r so that the circle will touch the vertex of the parabola.



Question 8 [10 marks]

Show that if $f(x)$ is a continuous function defined on $\mathbf{R}$ such that

$$f(f(x)) = x$$

for all $x \in \mathbf{R}$, then $f(x)$ has at least one fixed point.

END OF PAPER