NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 EXAMINATION 2008-2009

MA1101R Linear Algebra I

November 2008 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains a total of **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of the question.
3. Calculators are not allowed to be used. Students should show their working and solutions with all the necessary intermediate steps.

Question 1 [25 marks] For this question, let

$$\boldsymbol{u_1} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \boldsymbol{u_2} = \begin{pmatrix}-1\\1\\1\end{pmatrix}, \quad \boldsymbol{u_3} = \begin{pmatrix}3\\0\\4\end{pmatrix}, \quad \boldsymbol{v_1} = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix}2\\3\\4\end{pmatrix}.$$

(a) Let $\boldsymbol{A}$ be a 3×3 matrix. If $\boldsymbol{A} = [\boldsymbol{u_1}\ \boldsymbol{u_2}\ \boldsymbol{u_3}]$ (that is, $\boldsymbol{u_i}$ is the i-th column of $\boldsymbol{A}$), compute $\det(\boldsymbol{A})$.

(b) Explain why $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ is a basis for $\mathbb{R}^3$. If $\boldsymbol{w} = (x, y, z)^T$, find $(\boldsymbol{w})_S$.

(c) Write $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ as linear combinations of $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ and $\boldsymbol{u_3}$.

(d) Find all vectors orthogonal to $\text{span}\{\boldsymbol{u_3}, \boldsymbol{v_2}\}$.

(e) Find all vectors $\boldsymbol{y}$ such that $\text{span}\{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\} = \text{span}\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{y}\}$.

(f) Find an invertible matrix $\boldsymbol{B}$ such that **all** eigenvectors of $\boldsymbol{B}$ are scalar multiples of $\boldsymbol{u_1}$.

Question 2 [25 marks] For this question, let

$$\boldsymbol{A} = \begin{pmatrix}1 & 1 & 0\\-1 & a & 1\\-1 & -1 & 1\end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{pmatrix}, \quad \boldsymbol{C} = \begin{pmatrix}1 & \frac{1}{2} & \frac{1}{2}\\0 & 2 & \frac{1}{2}\\0 & 0 & 3\end{pmatrix}.$$

(a) If $a = -1$, find two different solutions to $\boldsymbol{AX} = \boldsymbol{B}$.

(b) If $a = -1$, $\boldsymbol{b} = \begin{pmatrix}1\\1\\1\end{pmatrix}$, find an $\boldsymbol{x}$ such that $||\boldsymbol{b} - \boldsymbol{Ax}||$ is minimized.

(c) Suppose $\boldsymbol{B}$ is the standard matrix of a linear transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$. Let l be the line

$$x = s, y = 0, z = 2s \text{ for } s \in \mathbb{R}$$

and $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the line l. Find the standard matrix for the linear transformation $T_1 \circ T_2$.

(d) If $a = 5$,

(i) show that 2 is an eigenvalue of $\boldsymbol{A}$;

(ii) find a basis for the eigenspace E_2.

(e) If $\boldsymbol{A}$ is the standard matrix of a linear transformation $T_3 : \mathbb{R}^3 \to \mathbb{R}^3$, find all values of a such that the range of T_3 is not $\mathbb{R}^3$.

(f) Prove that $\boldsymbol{B} - \boldsymbol{C}$ is diagonalizable.

Question 3 [25 marks] For this question, let

$$\boldsymbol{w_1} = \begin{pmatrix} 3 \\ 6 \\ 0 \end{pmatrix}, \quad \boldsymbol{w_2} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{w_3} = \begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{w_4} = \begin{pmatrix} 8 \\ 5 \\ -6 \end{pmatrix}.$$

(a) Explain why $S = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}, \boldsymbol{w_4}\}$ is a linearly dependent set.

(b) Show that $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is a basis for span(S).

(c) Prove that span$(S) = \mathbb{R}^3$.

(d) If $U = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ (standard basis for $\mathbb{R}^3$) and $V = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$, find the transition matrix from U to V.

(e) Compute the projection of $\boldsymbol{w_3}$ onto span$\{\boldsymbol{w_1}, \boldsymbol{w_2}\}$.

(f) Find a basis for span$\{\boldsymbol{w_1}, \boldsymbol{w_2}\} \cap$ span$\{\boldsymbol{w_3}, \boldsymbol{w_4}\}$.

Question 4 [25 marks] For this question, all matrices are symmetric matrices of order n and all vectors are column vectors. We also need the following definition

> A symmetric matrix $\boldsymbol{A}$ of order n is said to be **positive semidefinite** if for all nonzero $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} \geq 0$

(a) Prove that if $\boldsymbol{A}$ and $\boldsymbol{B}$ are both positive semidefinite matrices, then $\boldsymbol{A} + \boldsymbol{B}$ is also positive semidefinite.

(b) Prove that $\begin{pmatrix} 4 & 1 & 0 \\ 1 & 5 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ is positive semidefinite.

(c) If $\boldsymbol{A} = (a_{ij})$ is positive semidefinite, show that $a_{ii} \geq 0$ for $i = 1, 2, ..., n$.

(d) If $\boldsymbol{A} = (a_{ij})$ is positive semidefinite, show that $a_{ij}^2 \leq a_{ii}a_{jj}$ for all $i \neq j$.

END OF PAPER