QE ANALYSIS 2009 JAN

1.) $fg \geq 1$ a.e on E, so $g \geq 1/f$ a.e. on E.

$$\int_E f \cdot \int_E g = 1, \text{ so } \int_E f \cdot \int_E \left(\tfrac{1}{f}\right) + \left(g - \tfrac{1}{f}\right) = 1$$

$$\int_E f \cdot \int_E \frac{1}{f} \geq \left(\int_E \left(f \cdot \frac{1}{f}\right)^{1/2}\right)^2 = 1$$ By Cauchy-Schwarz inequality, but $\int_E f \cdot \int_E \frac{1}{f} \leq 1$, so $\int_E f \int_E \frac{1}{f} = 1$ and

$\int_E f \cdot \int_E (g - \frac{1}{f}) = 0$, which means that $\int_E g - \frac{1}{f} = 0$, i.e.

$g = \frac{1}{f}$ a.e. on E, so $fg = 1$ a.e. on E.

2.) By Lebesgue's Dominated Convergence Theorem, which can be applied because $f_1 \in L^1(E)$ and $|f_1|(x) \geq |f_k|(x)$ for $\forall x \in E$ and $\forall k$, we see that $f_n \to f$ a.e. on E implies that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

3.) $\phi\left(\int_{[0,1]} f\right) \leq \int_{[0,1]} \phi(f)$

By taking $f(x) = \begin{cases} a & \text{on } [0,\lambda] \\ b & \text{on } (\lambda, 1] \end{cases}$ for some $\lambda \in (0,1)$, we have

$$\phi(\lambda a + (1-\lambda) b) \leq \int_{[0,1]} \phi(f) = \lambda \phi(a) + (1-\lambda)\phi(b),$$ which

implies that ϕ is a convex function.

4.) On one hand, it is easy to see that

$$\frac{a_{n+1}}{a_n} = \frac{\int_E |f|^{n+1}}{\int_E |f|^n} \leq \frac{\int_E \|f\|_{L^\infty} \cdot |f|^n}{\int_E |f|^n} \leq \|f\|_{L^\infty}, \text{ so}$$

$\limsup \frac{a_{n+1}}{a_n} \leq \|f\|_{L^\infty}$.

On the other hand, let's denote by $A_\varepsilon \equiv \{x \in E : |f(x)| > \|f\|_{L^\infty} - \varepsilon\}$ (for some $\varepsilon > 0$).
Then $|A_\varepsilon| > 0$, and we can write

$$a_n = \int_E |f|^n = \underbrace{\int_{A_\varepsilon} |f|^n}_{b_n} + \underbrace{\int_{E\setminus A_\varepsilon} |f|^n}_{c_n}$$

But $b_n \ge \int_{A_{\varepsilon/2}} |f|^n \ge |A_{\varepsilon/2}| \cdot \left| \|f\|_{L^\infty} - \frac{\varepsilon}{2}\right|^n$, whereas

$c_n \le \int_{E\setminus A_\varepsilon} \left|\|f\|_{L^\infty} - \varepsilon\right|^n \le |E| \cdot \left|\|f\|_{L^\infty} - \varepsilon\right|^n$, so we can see that

$\lim_{n\to\infty} \frac{c_n}{b_n} = 0$. Therefore

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} \frac{b_{n+1}\left(1+\frac{c_{n+1}}{b_{n+1}}\right)}{b_n\left(1+\frac{c_n}{b_n}\right)} = \liminf_{n\to\infty} \frac{b_{n+1}}{b_n} =$$

$$= \liminf_{n\to\infty} \frac{\int_{A_\varepsilon} |f|^{n+1}}{\int_{A_\varepsilon} |f|^n} \ge \frac{\int_{A_\varepsilon} (\|f\|_{L^\infty}-\varepsilon)\cdot |f|^n}{\int_{A_\varepsilon} |f|^n} \ge \|f\|_{L^\infty} - \varepsilon \text{ for}$$

$\forall \varepsilon > 0$, therefore $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_{L^\infty}$.

5.) $\exp(-x^TAx + 2x^T V) = \exp(-(x-A^{-1}V)^T A (x-A^{-1}V) + V^TA^{-1}V)$

$$I \equiv \int_{\mathbb{R}^n} \exp(-x^TAx+2x^TV)\,dx = \int_{\mathbb{R}^n} \exp(-(x-A^{-1}V)^T A (x-A^{-1}V))\cdot \exp(V^TA^{-1}V)\,dx =$$

$= \left(\int_{\mathbb{R}^n} \exp(-x^TAx)\,dx\right)\cdot \exp(V^TA^{-1}V)$, since the translation by $A^{-1}V$ doesn't changes the integral. A is a pos. def matrix, therefore it can be written as $A = U^TDU = U^TD^{1/2}UU^TD^{1/2}U = A^{1/2}\cdot A^{1/2}$, and if we make the change of variable $y = A^{1/2}x$, then the Jacobi determinant is $\det(A^{1/2})$, so $dy = \det(A^{1/2})\,dx = \sqrt{\det A}\,dx$, and

$$I = \frac{1}{\sqrt{\det A}} \int_{\mathbb{R}^n} \exp(-y^Ty)\,dy \cdot \exp(V^TA^{-1}V).$$

By Tonelli's Theorem, $\int_{\mathbb{R}^n} \exp(-y^Ty) = \int_{\mathbb{R}^n} \exp(-\sum y_i^2) = \int_{y_1\in\mathbb{R}} \exp(-y_1^2)\cdot \int_{y_2\in\mathbb{R}} \exp(-y_2^2)\cdots \int_{\mathbb{R}} \exp(-y_n^2)\,dy_n$

$= \pi^{n/2}$, so the $I = \frac{\pi^{n/2}}{\ }\cdot \exp(V^TA^{-1}V)$

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6.) $\int_{-\pi}^{\pi} \frac{r}{re^{i\theta}+z}\,d\theta = \int_{\Gamma} \frac{r}{\xi+z}\,d\theta = \int_{\Gamma} \frac{r}{\xi+z}\,\frac{d\xi}{ire^{i\theta}} =$

$= \int_{\Gamma} \frac{r}{(\xi+z)\xi i}\,d\xi$, where Γ is the circle of radius r centered at 0.

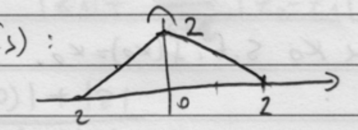
By the Residue Theorem, we can calculate this integral.
It has two residues, $z_1 = 0$, $z_2 = -z$, and if $|z| \neq 0$ and $|z| \neq r$, then
$\text{Res}\, z_1 = \lim_{\xi \to 0} \frac{r}{(\xi+z)\xi i}\cdot(\xi - 0) = \frac{r}{zi}$, $\text{Res}\, z_2 = \lim_{\xi \to -z} \frac{r}{(\xi+z)\xi}\cdot(\xi-(-z)) = -\frac{r}{zi}$, so

$$\int_{-\pi}^{\pi} \frac{r}{re^{i\theta}+z}\,d\theta = 2\pi i \sum_{z_i \in \text{int}\,\Gamma} \text{Res}(z_i) = \begin{cases} 0 & \text{if } z = 0 \\ 0 & \text{if } |z| < r \\ 2\pi \frac{r}{z} & \text{if } |z| > r \end{cases}$$

$$f(z) = \frac{1}{2\pi}\int_0^1 \int_{-\pi}^{\pi} \left(\frac{r}{re^{i\theta}+z}\right) d\theta\, dr = \begin{cases} |z| \le 1: & \int_0^{|z|} \frac{r}{z}\,dr = \frac{|z|^2}{2z} = \frac{\bar{z}}{2} \\ |z| \ge 1: & \int_0^1 \frac{r}{z}\,dr = \frac{1}{2z}. \end{cases}$$

7.) $\lim_{A \to \infty} \int_{-A}^{A} \left(\frac{\sin x}{x}\right)^2 e^{itx}\,dx$

First, we're going to show that $\left(\frac{\sin x}{x}\right)^2 \in L^1(\mathbb{R})$, because on $[-1,1]$, we know that $0 \le \left(\frac{\sin x}{x}\right)^2 \le 1$, and on $\mathbb{R} \setminus [-1,1]$, we have $\left(\frac{\sin x}{x}\right)^2 \le \frac{1}{x^2}$, so
$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 \le 2\left(1 + \int_1^{\infty} \frac{1}{x^2}\right) \le 4 < +\infty$.
Similarly, we can show that $\left(\frac{\sin x}{x}\right) \in L^2(\mathbb{R})$ too.
Then the integrand, $\left(\frac{\sin x}{x}\right)^2 \cdot e^{itx} \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ too.
For the characteristic function of the $[-1,1]$ interval, $\chi_{[-1,1]}$, we have $\int \chi_{[-1,1]}(s)\, e^{-ixs}\,ds = \left[\frac{e^{-ixs}}{-ix}\right]_{-1}^{1} = \frac{e^{ix} - e^{-ix}}{ix} = 2\frac{\sin(x)}{x}$, and by convolution we get
$\chi_{[-1,1]} * \chi_{[-1,1]}(s) = \int_x \chi_{[-1,1]}(x) \cdot \chi_{[-1,1]}(s-x)\,dx = \Lambda(s)$, where

$\Lambda(s)$: 

Then we have $\tilde{\Lambda}(s) = 4\left(\frac{\sin x}{x}\right)^2$, so by the inverse Fourier transformation, we get that
$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx}\,dx = \frac{\Lambda(t)}{4}$, so $\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx}\,dx = \frac{\pi \cdot \Lambda(t)}{2}$.

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8.) Let's make our notation more clear
$|\{f > \alpha\}| \equiv \omega_f(a)$, then

a.) If $f_k \nearrow f$, then $\{f_k > a\} \nearrow \{f > a\}$, so we must have $|\{f_k > a\}| \to |\{f > a\}|$.

b.) $f_k \to f$ in measure, so $|\{|f - f_k| > \varepsilon\}| \to 0$ as $k \to \infty$.

$\{f_k > a\} \subset \{f > a - \varepsilon\} \cup \{|f - f_k| > \varepsilon\}$, so

$|\{f_k > a\}| \le |\{f > a - \varepsilon\}| + |\{|f - f_k| > \varepsilon\}|$, by taking $\limsup_{k\to\infty}$, we have

$\limsup_{k\to\infty} |\{f_k > a\}| \le |\{f > a - \varepsilon\}|$.

Similarly

$\{f_k > a\} \cup \{|f - f_k| > \varepsilon\} \supset \{f > a + \varepsilon\}$, so
$|\{f_k > a\}| + |\{|f - f_k| > \varepsilon\}| \ge |\{f > a + \varepsilon\}|$, by taking $\liminf_{k\to\infty}$, we get the result:
$\liminf_{k\to\infty} |\{f_k > a\}| \ge |\{f > a + \varepsilon\}|$.

9.) Firstly, suppose there exist two fix points, x_0 and x_1, then
$\|f(x_0) - f(x_1)\| = \|x_0 - x_1\| \le \alpha \|x_0 - x_1\|$, which means $x_0 = x_1$, therefore at most 1 fix point exist.
Now let's chose an arbitrary y^0, and then let $y^{k+1} = f(y^k)$, for $k = 0, 1, 2, \ldots$. Then $\|y^{k+1} - y^k\| \le \alpha \|y^k - y^{k-1}\| \le \ldots \le \alpha^k \|y^1 - y^0\|$, therefore
for $l \le k$, $\|y^k - y^l\| \le \|y^1 - y^0\| \cdot \alpha^{l-1} \cdot (1 + \alpha + \alpha^2 + \ldots + \alpha^{k-l}) \le \|y^1 - y^0\| \cdot \frac{\alpha^{l-1}}{1-\alpha}$, so
y^k is a Cauchy series and by the completeness of $\mathbb{R}^n$ there must exist a limit x_0. Then for this limit, we have
$\|f(x_0) - y^{k+1}\| \le \|x_0 - y_k\|$, so $\|f(x_0) - x_0\| \le \|f(x_0) - y^{k+1}\| + \|y^{k+1} - x_0\| \le$
$\le \|y^{k+1} - x_0\| + \|y^k - x_0\| \to 0$ as $k \to \infty$, so $\|f(x_0) - x_0\| = 0$ and
$f(x_0) = x_0$
Therefore there exist a unique x_0 s.t. $f(x_0) = x_0$.

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10.) From complex Analysis we know that $g: D \to D$, and by choosing $a = f(0)$, we get $g(0) = 0$.

By applying the Schwartz Lemma, we get

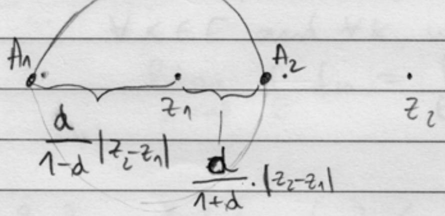
$$|g(z)| \le |z|$$

$$\frac{|f(z) - f(0)|}{|1 - \overline{f(0)} f(z)|} \le |z|$$

$$\frac{|f(z) - f(0)|}{|1/\overline{f(0)} - f(z)|} \le |f(0)| \cdot |z|$$

By denoting $C = |f(0)| \cdot |z|$, $f(0) = z_1$ and $1/\overline{f(0)} = z_2$, and $w = f(z)$, we get $\frac{|w - z_1|}{|w - z_2|} \le C$

The equation $\frac{|w - z_1|}{|w - z_2|} = d$ for some $d \ge 0$ is the equation of a circle called Appolonius circle (or a point for $d=0$). Now $C < 1$, so



For $1 > d_1 > d_2 \ge 0$ the circle for d_1 contains the circle for d_2, therefore $\frac{|w - z_1|}{|w - z_2|} \le C$ is a closed ball.

Now $z_1 = f(0)$ and $z_2 = 1/\overline{f(0)}$, so they lie on the same line as 0. So to estimate $|w|$, we only need to find the two intersects of the line (z_1, z_2) with the Appolonius circle. $|z_1| < 1$ and $|z_2| > 1$, so the distance of the two points A_1, A_2 from the origin of $d = C$ are (in the Oz_1 direction, so might be negative)

$$OA_1 = |z_1| - \frac{C}{1-C} \cdot |z_2 - z_1| = \left| |f(0)| - \frac{|f(0)||z|}{1 - |f(0)||z|} \cdot \left| \frac{1}{\overline{f(0)}} - f(0) \right| \right| =$$

$\frac{1}{|f(0)|} - |f(0)|$

$$= \left| \frac{|f(0)| - |f(0)|^2 |z|}{1 - |f(0)||z|} + \frac{|f(0)|^2 |z|}{1 - |f(0)||z|} - \frac{|z|}{1 - |f(0)||z|} \right| = \frac{|f(0)| - |z|}{1 - |f(0)||z|}.$$

$$OA_2 = |z_1| + \frac{C}{1+C} |z_2 - z_1| = |f(0)| + \frac{|f(0)||z|}{1 + |f(0)||z|} \cdot \left(\frac{1}{|f(0)|} - |f(0)| \right) =$$

$$= \frac{|f(0)| + |z|}{1 + |f(0)||z|}$$

Therefore we get $OA_1 \le |f(z)| \le OA_2$, which is what we needed to show.