NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

Qualification Examination Jan. 2009

Analysis

January, 2009 — Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper comprises **FOUR (4)** printed pages.
2. This paper consists of **TEN (10)** questions. Answer **ALL** of them.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Answer **all** *the questions in this paper*

Question 1 [10 marks]

Let E be a measurable set in R^n with its Lebesgue measure $|E| = 1$. Suppose f and g are both positive lebesgue measurable functions with $fg \geq 1$ a.e. on E. Show that, if $\int_E f \int_E g = 1$, then $fg = 1$ a.e. on E.

Question 2 [10 marks]

Suppose E is a Lebesgue measurable set in R^n. Let f_n be a sequence of the monotone decreasing positive measurable functions. Suppose f_n converges to a function f a.e. on E. Suppose $f_1 \in L^1(E)$. Show that $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Question 3 [10 marks]

Suppose ϕ is a real valued continuous function on R^1 such that

$$\phi(\int_{[0,1]} f) \leq \int_{[0,1]} \phi(f)$$

for every real simple measurable function f. Show that ϕ is a convex function.

Question 4 [10 marks]

Let E be a subset of R^n with $|E| < \infty$ in Lebesgue sense. Suppose $f \in L^\infty(E)$ and $\|f\|_{L^\infty} > 0$. Set

$$a_n = \int_E |f|^n$$

for $n = 1, 2, 3, \cdots$. Show that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \|f\|_{L^\infty}.$$

Question 5 [10 marks]

Let A be a real symmetric positive definite $n \times n$ matrix and V be a vector in R^n. Show that

$$\int_{R^n} \exp\{-x \cdot Ax + 2x \cdot V\}dx = \frac{\pi^{n/2}}{\sqrt{detA}} \exp\{V \cdot A^{-1}V\},$$

where $\cdot$ stands for the scalar product and A^{-1} means the inverse matrix of A. Notice that a matrix $A = (a_{ij})$ is called a real symmetric if each a_{ij} is a real number and $a_{ij} = a_{ji}$ for all $i, j = 1, 2, \cdots, n$. And A is positive definite if for all $x \neq 0$, $Ax \cdot x > 0$.

Question 6 [10 marks]

Define the complex valued function f by

$$f(z) = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} (\frac{r}{re^{i\theta} + z}) d\theta dr,$$

where $i = \sqrt{-1}$ and z is a complex number. Show that $f(z) = \bar{z}$ if $|z| < 1$ and that $f(z) = \frac{1}{z}$ if $|z| \geq 1$, where $\bar{\ }$ means complex conjugate.

Question 7 [10 marks]

For real number t, find the limit of

$$\lim_{A \to \infty} \int_{-A}^{A} (\frac{\sin x}{x})^2 e^{itx} dx.$$

Question 8 [10 marks]

If f is a measurable function on a measurable set E, define $\omega_f(a) = |\{x \in E | f(x) > a\}|$ for $a \in R$. If a sequence $\{f_k\}$ is monotone increasing and converges to f, show that $\omega_{f_k}(a)$ converges to $\omega_f(a)$. If $f_k \to f$ in measure as $k \to \infty$, show that $\limsup_{k \to \infty} \omega_{f_k}(a) \leq \omega_f(a - \epsilon)$ and $\liminf_{k \to \infty} \omega_{f_k}(a) \geq \omega_f(a + \epsilon)$ for every sufficiently small positive number ϵ.

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Question 9 [10 marks]

Let $f : R^n \longrightarrow R^n$ be a map such that for all $x, y \in R^n$,

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|$$

where $0 < \alpha < 1$ and $\|\cdot\|$ the standard norm on R^n. Show that there exists a unique point $x_0 \in R^n$ such that $f(x_0) = x_0$.

Question 10 [10 marks]

Suppose f is analytic in $D := \{z \in C | |z| < 1\}$ with $|f(z)| < 1$. By considering the function $g : D \longrightarrow D$ defined by

$$g(z) = \frac{f(z) - a}{1 - \overline{a} f(z)}$$

where $a = f(0)$, or otherwise show that

$$\frac{|f(0)| - |z|}{1 - |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)||z|}$$

for all $z \in D$.

END OF PAPER