2008-2009 SEM 1 ANALYSIS

1.) a.) Not true, for example let $X = Y = \mathbb{R}$, and $f(x) = x^2$, then $f(x)$ obviously not uniformly continuous, but if x_n is a Cauchy sequence, then the completeness of $\mathbb{R}$ implies $\exists x^* \in \mathbb{R}$ s.t. $\lim_{n\to\infty} x_n = x^*$, and the continuity of $f(x)$ implies that $f(x_n) \to f(x^*)$, so $f(x_n)$ is a Cauchy sequence too.

b.) $|f(x_0,y_0) - f(x,y)| = |f(x_0,y_0) - f(x,y_0) + f(x,y_0) - f(x,y)| =$
$= |\int_x^{x_0} f_x(s,y_0)\,ds + \int_y^{y_0} f_y(x,s)\,ds| \leq M(|x-x_0| + |y-y_0|)$ where M is the maximum of $|f_x|$ and $|f_y|$. This implies that if $(x,y) \to (x_0,y_0)$, then $f(x,y) \to f(x_0,y_0)$, so f is continuous.

c.) $$\int_0^1 \frac{1}{\sqrt{|x-r_n|}}\,dx \leq \int_{-1}^1 \frac{1}{\sqrt{|x|}}\,dx = \left[2\cdot x^{1/2}\right]_0^1 \cdot 2 = 4,$$

and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$, so

$$\int_0^1 \sum \frac{1}{n^2\sqrt{|x-r_n|}}\,dx \leq \frac{4\pi^2}{6} < +\infty,$$ this implies that

$$\sum \frac{1}{n^2\sqrt{|x-r_n|}} < +\infty$$ for a.e. $x \in [0,1]$.

2.) f is A.C. means that for $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if we take $[a_1,b_1], [a_2,b_2] \ldots$ intervals with $\sum(b_i - a_i) < \delta$, then $\sum |f(b_i) - f(a_i)| < \varepsilon$.
If V is A.C., then for $\forall \varepsilon > 0$ $\exists \delta_V > 0$ s.t. $\sum |V(b_i) - V(a_i)| < \varepsilon$ for all $[a_1,b_1], [a_2,b_2] \ldots$ intervals with $\sum(b_i - a_i) < \delta_V$. But $|f(b_i) - f(a_i)| \leq V(b_i) - V(a_i)$, so by choosing $\delta = \delta_V$, we get that f is A.C. too.

3.) Thm 5.44 of Measure and Integral: An introduction to real analysis.

4.) $$F(x) = \int_0^x f(x+t)\,dt = \int_x^{2x} f(t)\,dt = \int_0^{2x} f(t)\,dt - \int_0^x f(t)\,dt$$

By Fundamental Theorem of calculus, we get
$F'(x) = 2\cdot f(2x) - f(x)$
On the other hand, $\int_0^a f'(a+t)\,dt = \int_a^{2a} f'(t)\,dt = f(2a) - f(a)$, therefore
$F'(a) = 2\cdot f(2a) - f(a) = f(2a) + \int_0^a f'(a+t)\,dt$.

5.) Let $K = \sup_n \int_0^1 |f_n| < +\infty$.

By Fatou's Lemma applied to $|f_K|$, we have
$$\int_0^1 \liminf |f_K| \leq \liminf \int_0^1 |f_K|$$
$$\int_0^1 |f| \leq \liminf \int_0^1 |f_n| \leq K < +\infty,$$ so f is integrable on $[0,1]$.

Let's define $M_N(x) = \max(\min(x, N), -N)$, and $f_n^N(x) = M_N(f_n(x))$, $f^N(x) = M_N(f(x))$. Then $M_N(x)$ is continuous, therefore f_n^N and f^N are measurable functions, and bounded by N, so they're integrables too. Moreover
$$\lim_{n\to\infty} \left|\int_0^1 f - f_n\right| \leq \underbrace{\left|\int_0^1 f - f^N\right|}_{(I)} + \underbrace{\left|\int_0^1 f^N - f_n^N\right|}_{(II)} + \underbrace{\left|\int_0^1 f_n^N - f_n\right|}_{(III)}$$

If $f \geq 0$, then $f^N \nearrow f$, so by monotone convergence theorem (I) $\to 0$ as $N \to +\infty$. Generally we can write $f = f^+ - f^-$, $f^N = f^{N+} - f^{N-}$, and get the same.

For (II), we can apply Bounded Convergence theorem, so (II) $\to 0$ as $n \to \infty$.

For (III), we can write
$$\left|\int_{[0,1]} f_n^N - f_n\right| \leq \int_{[0,1]} |f_n^N - f_n| \leq \int_{\{|f_n| > N\}} |f_n^N - f_n| \leq \int_{\{|f_n| > N\}} |f_n|$$

But $\int_{[0,1]} |f_n| < K$, so $|\{|f_n| > N\}| < K/N$. By (b), for $\forall \varepsilon > 0$, we have a δ, such that if $|E| < \delta$ than $\int_E |f_n| < \varepsilon$, and by choosing N large enough s.t. $\frac{K}{N} < \delta$, we get (III) $< \varepsilon$.

Therefore by taking the limit in $n \to \infty$, we get for large N

$\lim_{n\to\infty} \left|\int f_n - \int f\right| \leq$ (I) $+ \varepsilon$, and by taking $N \to \infty$, (I) $\to 0$, so

$\lim_{n\to\infty} \left|\int f_n - \int f\right| < \varepsilon$, since ε is arbitrary, we get

$$\lim_{n\to\infty} \int_{[0,1]} f_n = \int_{[0,1]} f.$$