NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2008-2009

Ph.D. QUALIFYING EXAMINATION

PAPER 2

ANALYSIS

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

Ph.D. Qualifying Examination

Year 2008–2009, Semester I

Analysis

Part 1. (65 marks)

1. **[5 points each]** Each of the following statements is either **TRUE** or **FALSE**. Prove the true statements and give counterexamples to the false statements.

   (a) Let X and Y be metric spaces. A function $f : X \to Y$ is uniformly continuous on X if and only if it maps Cauchy sequences in X onto Cauchy sequences in Y.

   (b) If f is a real-valued function defined on $\mathbb{R}^2$ such that f_x and f_y exist on $\mathbb{R}^2$ and are bounded there, then f is continuous on $\mathbb{R}^2$.

   (c) Let $(r_n)_{n=1}^{\infty}$ be an arbitrary sequence of numbers in $[0,1]$. The series
$$\sum_{n=1}^{\infty} \frac{1}{n^2\sqrt{|x-r_n|}}$$
   converges for almost all x in $[0,1]$.

2. **[10 points]** Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation on $[a,b]$. For any $x \in [a,b]$, let $V(x)$ be the variation of f on $[a,x]$. Show that if V is absolutely continuous on $[a,b]$, then so is f.

3. **[10 points]** Let f_1 and f_2 be nonnegative Lebesgue measurable functions on $\mathbb{R}$. Suppose that the sets $\{x : f_1(x) > a\}$ and $\{x : f_2(x) > a\}$ are equal in measure for all $a > 0$. Prove that f_1 is Lebesgue integrable if and only if f_2 is Lebesgue integrable; in which case, show that $\int f_1 = \int f_2$.

4. **[15 points]** Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $\mathbb{R}$ and assume that f' is continuous on $\mathbb{R}$. Define $F : \mathbb{R} \to \mathbb{R}$ by
$$F(x) = \int_0^x f(x+t)\, dt.$$
   Show that F is differentiable on $\mathbb{R}$ and that, for all $a \in \mathbb{R}$,
$$F'(a) = f(2a) + \int_0^a f'(a+t)\, dt.$$

5. **[15 points]** Let $(f_n)_{n=1}^{\infty}$ be a sequence of Lebesgue integrable functions on $[0,1]$ such that

   (a) $\sup_n \int_0^1 |f_n| < \infty$,

   (b) For all $\epsilon > 0$, there exists $\delta > 0$ so that $\sup_n \int_E |f_n| < \epsilon$ for every measurable subset E of $[0,1]$ with $|E| < \delta$.

   Show that if $(f_n)_{n=1}^{\infty}$ converges almost everywhere on $[0,1]$ to a function f, then f is integrable on $[0,1]$ and $\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$.
   (It may be helpful to consider the functions $\max(\min(f_n, N), -N)$ for $N \in \mathbb{N}$.)

Part 2 (35 marks)

6. **[13 marks]**

(a) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function in the complex plane $\mathbb{C}$. Let z_o be a point in $\mathbb{C}$, and let C be the unit circle $|z - z_o| = 1$ centered at z_o and oriented in the counter-clockwise direction. It is given that $f(z) \neq f(z_o)$ for all $\underline{z \text{ inside or on } C \text{ except } z_o}$,

$$f'(z_o) = 2, \quad f''(z_o) = 3 \quad \text{and} \quad \int_C \frac{f'(z)}{f(z) - f(z_o)}\, dz = 2\pi i.$$

Evaluate the integral $\displaystyle\int_C \frac{1}{(f(z) - f(z_o))^2}\, dz$. Justify your answer.

(b) Solve the equation $4 + \cos z = 2\sinh(iz)$. Express your answers in Cartesian form.

7. **[12 marks]** Let $D := \{z \in \mathbb{C} : |z| < 2\}$ denote the disc of radius 2 and centered at the origin in the complex plane $\mathbb{C}$. Suppose the function $f : D \setminus \{\frac{i}{2}\} \to \mathbb{C}$ is analytic in $D \setminus \{\frac{i}{2}\}$, and f has a simple pole at the point $z = \frac{i}{2}$. Let $\sum_{n=0}^{\infty} a_n z^n$ denote the Maclaurin series of f. It is also given that $a_n \neq 0$ for all $n \geq 0$. Is it true that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = -2i\,?$$

Justify your answer.

8. **[10 marks]** Let $f : \mathbb{C} \to \mathbb{C}$ be a <u>non-constant</u> entire function in the complex plane $\mathbb{C}$ such that $f(z + i) = f(z)$ for all $z \in \mathbb{C}$. Let U be an open subset of $\mathbb{C}$, and let $z_o \in U$. Suppose $g : U \setminus \{z_o\} \to \mathbb{C}$ is an analytic function on $U \setminus \{z_o\}$. It is given that z_o is not a removable singularity of g. Is it true that $f \circ g$ has an essential singularity at z_o? Justify your answer.

END OF PAPER