NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2007-2008

MA3501 Mathematical Methods in Engineering

April/May 2008 – Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **SEVEN(7)** printed pages, including three appendices: Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Three Appendices (P5-P7): Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

Question 1 [14 marks]

(a) The Department of Agriculture reports that the mean cost of raising a child from birth to age 2 in a rural area is \$8390. You believe this value is incorrect, so you select a random sample of 900 children (age 2) and find that the mean cost is \$8275 with a standard deviation of \$1540. At 5% level of significance, is there enough evidence to conclude that the mean cost is different from \$8390?

(b) In an advertisement, a pizza shop claims that its mean delivery time is less than 30 minutes. A random selection of 36 delivery times has a sample mean of 31.5 minutes and a standard deviation of 3.5 minutes. Is there enough evidence to support the claim at 1% level of significance?

Question 2 [14 marks]

(a) Solve the following heat equation by the method of separation of variables:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) &&, \ 0 < x < 1,\ t > 0 \\ u_x(0,t) = 0,\ u_x(1,t) &= 0 &&, \ t > 0 \\ u(x,0) &= x &&, \ 0 < x < 1 \end{aligned}$$

(b) Prove that for $0 < x < 1$

$$x = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos[(2n-1)\pi x]$$

Question 3 [14 marks]

Use the method of the Fourier transform to find the solution of the following PDE:

$$\begin{aligned} u_{tt}(x,t) &= a^2 u_{xx}(x,t) &&, \ -\infty < x < \infty,\ t > 0 \\ u(x,0) &= 0 &&, \ -\infty < x < \infty \\ u_t(x,0) &= f(x) &&, \ -\infty < x < \infty \end{aligned}$$

Suppose $f(x) = \cos x$ and $a = 1$. Find the value of $u(\pi, \frac{\pi}{2})$.

Question 4 [14 marks]

Solve, by the method of characteristics, the following 1^{st} order PDE:

$$e^{-t}u_t(x,t) + u_x(x,t) = xe^{-t}, \ u(x,0) = x$$

Question 5 [14 marks]

Suppose that the temperature $u(x,t)$ satisfies the following heat equation:

$$\begin{aligned} u_t(x,t) &= 4u_{xx}(x,t) &&, \ 0 < x < 2, \ t > 0 \\ u_x(0,t) &= 2[u(0,t) - 2] &&, \ t > 0 \\ u(2,t) &= -3 &&, \ t > 0 \\ u(x,0) &= \sin\tfrac{\pi x}{2} &&, \ 0 < x < 2 \end{aligned}$$

Find the steady-state (final) temperature $v(x)$.

Question 6 [16 marks]

(a) Let γ be a circle with centre i and radius 2 in anticlockwise direction.

Evaluate the following integrals:

(i) $\displaystyle\int_\gamma \frac{1}{z(z^2+4)}\, dz$

(ii) $\displaystyle\int_\gamma \frac{1}{(z-4)}\, dz$

(b) (i) Find the image of the upper half of the unit disk with centre 0 under the mapping $w_1 = i\dfrac{1-z}{1+z}$.

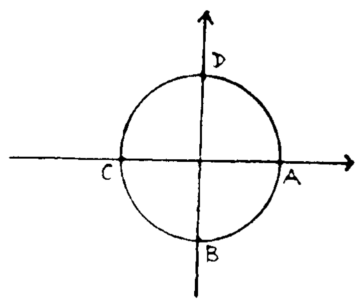
(ii) Find the image of the upper half of the unit disk with centre 0 under the mapping $w_2 = -\left(\dfrac{1-z}{1+z}\right)^2$.

Question 7 [14 marks]

(a) Let G be the upper half-plane without x-axis, i.e., $G = \{w : w = u + iv, v > 0\}$. Let $\Phi(w) = a\text{Arg }(w-1) - b\text{Arg } w$, where a and b are real constants. Prove that $\Phi(w)$ is a solution of Laplace's equation

$$\Phi_{uu}(w) + \Phi_{vv}(w) = 0 \text{ in } G.$$

(b) Let E be an open unit disk with centre $(0, 0)$



Solve the following PDE:

$$\begin{aligned} \phi_{xx}(z) + \phi_{yy}(z) = 0 &\quad \text{in } E \\ \phi(z) = 0 &\quad \text{on the boundary } ABCD \\ \phi(z) = 100 &\quad \text{on the boundary } AD \end{aligned}$$

(You may use the following fact without proof $w = i\dfrac{1-z}{1+z}$ maps E onto G given in (a) and the boundary of E onto the boundary of G.)

END OF PAPER

Formulae

(A) $\dfrac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)x'(t)$

(B) $\displaystyle\int_0^1 x\cos n\pi x \; dx = [(-1)^n - 1]\,\frac{1}{n^2\pi^2}, n = 1, 2, \ldots$

(C) $\text{Log} z = \ln|z| + i\text{Arg } z,\ -\pi < \text{Arg } z < \pi$

(D) Fourier Transforms

Let $\mathcal{F}(f)(\omega) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx,\ -\infty < \omega < \infty.$

$\mathcal{F}(f(x))$ denoted by $\hat{f}(\omega)$.

$\mathcal{F}(u(x,t))$ denoted by $\hat{u}(\omega,t)$.

$\mathcal{F}(u_t(x,t)) = \dfrac{d}{dt}\hat{u}(\omega,t)$.

$\mathcal{F}(u_{tt}(x,t)) = \dfrac{d^2}{dt^2}\hat{u}(\omega,t)$.

$\mathcal{F}(u_{xx}(x,t)) = -\omega^2\hat{u}(\omega,t)$.

$\mathcal{F}\left[\dfrac{\sqrt{2\pi}}{2a}\chi_{(-at,at)}(x)\right](\omega) = \dfrac{\sin a\omega t}{a\omega}$, where $\chi_{(-at,at)}(x) = \begin{cases} 1, & \text{if } x \in (-at, at), \\ 0, & \text{otherwise .}\end{cases}$

$\mathcal{F}^{-1}\left(\hat{f}(\omega)\hat{g}(\omega)\right) = (f * g)(x)$

$(f * g)(x) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} f(x-\tau)g(\tau)\,d\tau$