NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2007-2008

MA5205 Graduate Analysis I

November/December 2007— Time allowed : 2 and 1/2 hours

INSTRUCTIONS TO CANDIDATES

(1) This examination paper consists of **TWO** (2) sections: Section A and Section B. It contains a total of **SEVEN** (7) questions and comprises **FOUR** (4) printed pages.

(2) Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

(3) Answer not more than **TWO** (2) questions from **Section B**. Each question in Section B carries 18 marks.

(4) Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

(5) All theorems and results used in your answers should be clearly stated.

SECTION A
Answer **all** the questions in this section. Section A carries a total of 64 marks.

Question 1 [30 marks]

Prove or disprove each of the following statements.

(i) If f is a nonnegative measurable function on $\mathbb{R}^n$, then there exists a sequence of functions $\{f_k\} \subset \hat{S} = \{\sum_{i=1}^n a_i \chi_{E_i} : n \in \mathbb{N}, 0 \le a_i < \infty, |E_i| < \infty \text{ for all } i\}$ such that $\lim_{k\to\infty} f_k(x) = f(x)$ and $f_k(x) \le f_{k+1}(x) \le f(x)$ for all $x \in \mathbb{R}^n$.

(ii) If f is a function of bounded variation on $[a,b]$, then it is Borel measurable on $[a,b]$.

(iii) If ϕ is an additive set function (signed measure) on a measurable space $\langle X, \Sigma\rangle$ and $f : X \to \mathbb{R}$ is a measurable function, then given any $\varepsilon > 0$, there exist $M > 0$ and $E \subset X$ such that $|f(x)| \le M$ for $x \in E$ and $|\phi(X\backslash E)| < \varepsilon$.

(iv) If $f \in L^p(\mathbb{R}^n)$ where $1 \le p < \infty$, then given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E |f| < \varepsilon$ whenever $|E| < \delta$.

(v) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f_r(x) = f(x) + r$ is Borel measurable for all positive rational numbers r. Then f is also Borel measurable.

(vi) Let $f_n, f \in L^2(\mathbb{R})$ for all n such that $f_n \to f$ almost everywhere. If $\int f_n g \to \int fg$ for all $g \in L^2(\mathbb{R})$, then $f_n \to f$ in $L^2(\mathbb{R})$.

Question 2 [15 marks]

Let $P_y(x) = P(x,y) = \frac{1}{\pi}\frac{y}{y^2+x^2}$. Suppose $f(x_0^+) = \lim\limits_{x\to x_0^+} f(x)$ and $f(x_0^-) = \lim\limits_{x\to x_0^-} f(x)$ are both finite. If $f \in L^\infty(\mathbb{R})$,

(i) for any $y > 0$, show that

$$f*P_y(x_0) - \frac{f(x_0^+) + f(x_0^-)}{2} = \int_0^\infty (f(x_0-t) - f(x_0^-))P_y(t)dt + \int_{-\infty}^0 (f(x_0-t) - f(x_0^+))P_y(t)dt;$$

(ii) hence or otherwise, show that

$$\lim_{y\to 0^+} f * P_y(x_0) = \frac{f(x_0^+) + f(x_0^-)}{2}.$$

Question 3 [7 marks]

Suppose f and w are nonnegative measurable functions on $\mathbb{R}^n$ such that $\displaystyle\int f(x)^p w(x)\,dx < \infty$ and $\displaystyle\int f(x)^2 w(x)\,dx < \infty$, where $0 < p < 2$. Show that $\displaystyle\int f(x)^r w(x)\,dx < \infty$ for any $p < r < 2$.

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Question 4 [12 marks]

Let f^* be the Hardy-Littlewood maximal function of f.

(i) Show that $|f(x)| \leq f^*(x)$ for almost all x if $f \in L^2(\mathbb{R}^n)$.

(ii) If $f_n \to f$ in $L^2(\mathbb{R}^n)$, show that $f_n^* \to f^*$ in $L^2(\mathbb{R}^n)$.

SECTION B

Answer not more than **two** questions in this section. Each question in this section carries 18 marks.

Question 5 [18 marks]

(a) Let E be a measurable set in $\mathbb{R}^n$ and $1 < r < \infty$. If $\|g\|_{L^p(E)} \leq M$ for all $1 \leq p < r < \infty$, show that $\|g\|_{L^r(E)} \leq M$.

(b) Let $D^+f(x) = \lim_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h}$. If f is an absolutely continuous function on an interval $[a, b]$ such that $D^+f = 0$ almost everywhere on (a, b), show that f must be a constant function.

Question 6 [18 marks]

(a) Let $\langle X, \Sigma, \mu \rangle$ be a measure space. Let $\{f_n\}$ be a sequence of measurable functions that converges to a function f in μ. Suppose there exists a sequence of integrable functions $\{g_n\}$ that converges to an integrable function g almost everywhere. If $|f_n| \leq g_n$ except on a set of μ-measure 0 for all n and

$$\lim_{n \to \infty} \int g_n = \int g,$$

show that $\lim_{n \to \infty} \int f_n = \int f$.

(b) Let f be a real-valued measurable function on a measure space $\langle X, \Sigma, \mu \rangle$ with $\mu(X) < \infty$. If there exists a constant $C > 0$ such that

$$\mu\{x \in X : |f(x)| > t\} \leq C/t^2 \quad \text{for all } t > 0,$$

show that $f \in L^p(X, d\mu)$ for all $0 < p < 2$.

Question 7 [18 marks]

(a) Let $f(x,y)$ be an integrable function in $\mathbb{R}$ for each $y \in \mathbb{R}$. Suppose for each $y > 0$, $\frac{\partial}{\partial y} f(x,y)$ exists and $|\frac{\partial}{\partial y} f(x,y)| \le \frac{y}{x^2+y^2}$. Show that the function

$$g(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

is differentiable for all $y > 0$ and

$$g'(y) = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} f(x,y)dx.$$

(b) Let f^* be the Hardy-Littlewood maximal function of f and let $w(x)$ be a nonnegative function on $\mathbb{R}^n$. Suppose there exist constants $C_1, C_2 > 0$ such that

$$\int_{f^*>t} w(x)dx \le C_1 \frac{\int |f(x)|w(x)dx}{t} \quad \text{for all } t > 0$$

and

$$\int (f^*(x))^2 w(x)dx \le C_2 \int |f(x)|^2 w(x)dx$$

for all locally integrable functions f. Show that for each $1 < p < 2$, there exists a constant $C_3 > 0$ such that

$$\int (f^*(x))^p w(x)dx \le C_3 \int |f(x)|^p w(x)dx$$

for all locally integrable functions f.

END OF PAPER