NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2007-2008

MA5203 Graduate Algebra I

December 2007 — Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This is a closed book examination.

2. This examination paper contains a total of **NINE (9)** questions and comprises **THREE (3)** printed pages.

3. Answer **ALL** questions. The marks for the questions are not necessarily the same; marks for each question are indicated at the beginning of the question.

4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1. [10 marks]

Show that a finite group of order 105 has a non-trivial normal subgroup.

Question 2. [10 marks]

Let p be a prime. Determine all groups of order p^2.

Question 3. [10 marks]

Let R be a commutative ring with 1. Assume that R satisfies the ascending chain condition. Let I be an ideal generated by an infinite sequence of elements $x_1, x_2, \ldots$ in R. Show that I is finitely generated.

Question 4. [10 marks]

Let A be a rational 3×3 matrix such that $A^3 = A$. Show that A can be diagonalized.

Question 5. [20 marks]

Let $\zeta_8 = e^{\frac{i\pi}{4}}$.

(i) Show that $\Phi_8(x) = x^4 + 1$ is irreducible in $\mathbb{Q}[x]$.

(ii) Determine the Galois group of $K = \mathbb{Q}(\zeta_8)$ over $\mathbb{Q}$.

(iii) Determine all quadratic extensions of $\mathbb{Q}$ contained in K.

Question 6. [10 marks]

Compute the Galois group of the splitting field of the polynomial $x^3 - 2 \in F[x]$ where F is

(i) $\mathbb{R}$.

(ii) $\mathbb{Q}$.

(iii) $\mathbb{F}_5$.

(iv) $\mathbb{F}_7$.

Question 7. [10 marks]

Find a real number α such that $\mathbb{Q}(\alpha)$ is a Galois extension of $\mathbb{Q}$ with the Galois group $\mathbb{Z}/5\mathbb{Z}$.

Question 8. [10 marks]

Compute the following tensor products of $\mathbb{Z}$-modules.

(i) $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$.

(ii) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Question 9. [10 marks]

Let $Q(x, y, z) = x^2 + y^2 - 2z^2$. Find $(x, y, z) \in \mathbb{Q}^3$ such that

$$Q(x, y, z) = 2007.$$

END OF PAPER