Ph.D. Qualifying Examination

Linear Algebra

Semester 2, 2007-2008

Answer all questions.

1. Let V be a finite dimensional vector space and W a subspace of V. Prove that

$$\dim V/W = \dim V - \dim W.$$

Here V/W is the quotient space of V by W.

2. Let V be a finite dimensional complex vector space and $T : V \to V$ a linear operator on V. Prove that V has a basis $\mathcal{B}$ such that the matrix $[T]_{\mathcal{B}}$ is upper triangular.

3. Let $\mathbb{F}$ be a field and let J be a matrix over $\mathbb{F}$ which is in Jordan canonical form. Prove that J is similar to its transpose J^t over $\mathbb{F}$.

4. Let V be a finite-dimensional real vector space and let $B : V \times V \to \mathbb{R}$ be a symmetric bilinear form.

   (i) Let $q(\mathbf{v}) = B(\mathbf{v}, \mathbf{v})$, $\mathbf{v} \in V$. Prove that

   $$B(\mathbf{u}, \mathbf{v}) = \frac{1}{2} q(\mathbf{u} + \mathbf{v}) - q(\mathbf{u}) - q(\mathbf{v})$$

   for all $\mathbf{u}, \mathbf{v} \in V$.

   (ii) Prove that V has a basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ such that $B(\mathbf{v}_i, \mathbf{v}_j) = 0$ for all $1 \leq i, j \leq n$ with $i \neq j$.

5. Let V be a finite dimensional real inner product space and let S and T be self-adjoint operators on V.

   (i) Prove that if the subspace W of V is T-invariant, then its orthogonal complement $W^{\perp}$ is also T-invariant.

   (ii) Prove that if $ST = TS$, then V has an orthonormal basis $\mathcal{B}$ such that both the matrices $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ are diagonal.

END OF PAPER