National University of Singapore

Ph.D. Qualifying Examination
Year 2007–2008 Semester II

Algebra

Answer all questions. Each question carries 25 marks.

(1) Let R be a commutative ring with 1.

(a) Let I be an ideal of R. Explain briefly what is meant to say that (i) I is *prime*, (ii) I is *maximal*.

(b) Prove or disprove each of the following statements:

(i) If I is a maximal ideal of R, then I is prime.

(ii) If I is a nonzero prime ideal of R, then I is maximal.

(2) Let p and q be a prime integers with $p \leq q$.

(a) Show that any group of order pq has a normal subgroup of order q.

(b) Hence, or otherwise, classify all groups of order pq up to isomorphism.

(3) Let n be a fixed positive integer.

(a) Prove that $\mathbb{Q}(\cos \frac{2\pi i}{n})$ is an algebraic extension over $\mathbb{Q}$.

(b) Determine the degree $[\mathbb{Q}(\cos \frac{2\pi i}{n}) : \mathbb{Q}]$.

(4) Let R be a ring with 1, and let M be a left R-module. Prove that the following statements are equivalent:

(a) M is nonzero, and if N is a submodule of M, then $N = 0$ or $N = M$.

(b) For every $m \in M \setminus \{0\}$, $M = \{rm \mid r \in R\}$.

(c) There exists a maximal left ideal I of R such that $M \cong R/I$ as left R-modules.