PhD Qualifying Examination

Linear Algebra

Sem 1, 2007/2008

(1) Let $f_1(x), f_2(x), \cdots, f_n(x)$ be functions having derivatives of all orders and let $F(x)$ be the function obtained by taking the determinant of

$$\begin{pmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix}.$$

Calculate $F'(x)$ (please express your answer as an $n \times n$ matrix). Hence or otherwise, prove that if $f_1(x), f_2(x), \cdots, f_n(x)$ are polynomials of degrees $n-1$ or less. Then $F'(x) = 0$.

(ii) Let A and B be two matrices with integral entries. Suppose that A and B admit the same characteristic polynomial $f(x)$. Suppose further that $f(x)$ is irreducible over $\mathbb{Z}$. Determine whether A and B are similar to each other over $\mathbb{C}$. Justify your answer.

(iii) Let V be a vector space over $\mathbb{R}$. A *scalar product* on V is a function $\langle \, | \, \rangle : V \times V \to \mathbb{R}$ such that (i) $\langle u|v \rangle = \langle v|u \rangle$ for all $u, v \in V$, (ii) $\langle u|v + w \rangle = \langle u|v \rangle + \langle u|w \rangle$ for all $u, v, w \in V$, (iii) $\langle cu|v \rangle = c \langle u|v \rangle$ for all $c \in \mathbb{R}$, $u, v \in V$. Let V be a scalar product space of finite dimensional over $\mathbb{R}$. Prove that V has an orthogonal basis. Determine whether V has an orthonormal basis. Justify your answer.

(iv) Let V be the vector space of $n \times n$ matrices over $\mathbb{C}$ and let A be an element of V. Let $f : V \to V$ be the map defined by $f(X) = AX$. Show that f is linear and that $\det f = (\det A)^n$.

(v) Let $\tau \in \mathbb{C}$ and let $V_\tau = \{\sum_{i=0}^{n} a_i \tau^i \, : \, a_i \in \mathbb{Q}\}$. Prove or disprove the following :

(a) V_τ is a finite dimensional vector space over $\mathbb{Q}$ if and only if $f(\tau) = 0$ for some $f(x) \in \mathbb{Q}[x] - \{0\}$.

(b) Suppose that V_τ and V_ν are finite dimensional. Then $V_{\tau+\nu}$ and $V_{\tau\nu}$ are finite dimensional.

Justify your answer.