Ph.D. Qualifying Examination (August 2007)
Complex Analysis

ANSWER ALL 6 QUESTIONS.

Convention: *Throughout this paper,* $\mathbf{R}$ *denotes the set of real numbers, and* $\mathbf{C}$ *denotes the set of complex numbers. For* $z \in \mathbf{C}$, $\operatorname{Re} z$ *denotes the real part of* z, $\operatorname{Im} z$ *denotes the imaginary part of* z*. In addition, for a complex variable* $z = x + iy$, x *denotes the real part of* z*, and* y *denotes the imaginary part of* z*, unless otherwise stated.*

1. (a) Determine all the complex numbers z such that $\cos z$ is purely imaginary.

   (b) Let $D := \{z \in \mathbf{C} \,\big|\, |z| < 1\}$, and suppose $f : D \to \mathbf{C}$ is an analytic function such that its derivative f' is injective on D. Prove that $f''(z) \neq 0$ for all $z \in D$.

2. Use the Cauchy residue theorem to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x + \sin 3x}{x^2 + 25}\, dx.$$

   Justify your steps.

3. Consider the punctured disk $D := \{z \in \mathbf{C} \,\big|\, 0 < |z| < 1\}$. Suppose $f : D \to \mathbf{C}$ is an analytic function such that

$$|f''(z)| \leq \frac{2}{|z|^2} \quad \text{for all } z \in D.$$

   Is it true that f has a removable singular point at $z = 0$? Justify your answer.

4. Find a conformal map from the domain

$$D_1 := \{z = x + iy \in \mathbf{C} \,\big|\, |x + y| < 1\}$$

   onto the second quadrant $D_2 := \{z = x + iy \in \mathbf{C} \,\big|\, x < 0 \ \text{ and } \ y > 0\}$.

5. Determine all the entire functions f such that the real part of f satisfies the inequality

$$\operatorname{Re} f(z) \leq 2\,\big|\ell n\,|z|\,\big| + 3 \quad \text{for all } z \in \mathbf{C}.$$

   Here ℓn denotes the (real-valued) natural logarithmic function.

6. Consider the unit disk $D := \{z \in \mathbf{C} \,\big|\, |z| < 1\}$. Suppose $f : D \to \mathbf{C}$ is an analytic function such that

$$f(z) \in D \qquad \text{for all } z \in D,$$

   and f is not one-to-one on D. Prove that $|f'(0)| < 1$.