Ph.D. Qualifying Examination

Analysis

Sem 1, 2007/2008

1. (a) A subset K of a metric space X is said to be compact if every open conver of K contains a finite subcover.

   Prove that compact subsets of metric spaces are closed.

   (b) A subset E of a metric space X is said to be perfect if E is closed and if every point of E is a limit point of E.

   Prove that if E is a non-empty perfect set of $\mathbb{R}$. Then E is uncountable.

   (c) Prove that the open interval (a, b) is uncountable.

2. (a) Let $f_n : \mathbb{R} \to \mathbb{R}$ be continuous, $n = 1, 2, ...$ Suppose that

$$f(x) = \sum_{u=0}^{\infty} f_n(x) \quad \text{exists for every} \quad x \in \mathbb{R}.$$

   Is $f : \mathbb{R} \to \mathbb{R}$ continuous ? Justify your answer.

   (b) Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E and suppose that

$$\lim_{t \to x} f_n(t) = A_n, \quad n = 1, 2, \dots .$$

   Prove that $\lim_{n\to\infty} A_n$ exists and

$$\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t).$$

   (c) Let $\{f_n\}$ be a sequence of continuous functions on $(0, 1)$ such that $\{f_n\}$ converges pointwise to a continuous function on $(0, 1)$ and $f_n(x) \geq f_{n+1}(x)$ for all $x \in (0, 1)$, $n = 1, 2, \dots$.

   Does $\{f_n\}$ converge uniformly to f on $(0, 1)$? Justify your answer.

3. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$, let $\alpha > 0$. If there exists a constant $k > 0$ such that

$$|f(x) - f(y)| \leq k|x - y|^{\alpha}$$

for all x, $y \in A$, then f is said to be a Lipschitz function of order α on A.

(a) Suppose f is a Lipschitz function of order α on $(0, 1)$ where $\alpha > 1$. Prove that f is differentiable on $(0, 1)$ and find its derivative f'.

(b) Give an example of a Lipschitz function of order $\frac{1}{2}$ but not of order 1 on $[0, 1]$.

(c) Is every uniformly continuous function on $[0, 1]$ is a Lipschitz function of order 1? Justify your answer.

4. (a) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots$. Suppose $\{f_n\}$ converges uniformly on $[0, 1]$. Prove that $\{f_n\}$ is equicontinuous on $[0, 1]$.

(b) Let $f_n : [0, 1] \to \mathbb{R}$ be continuous, $n = 1, 2, \ldots.$ Suppose $\{f_n\}$ is pointwise bounded and equicontinuous on $[0, 1]$. Prove that (i) $\{f_n\}$ is uniformly bounded on $[0, 1]$; (ii) $\{f_n\}$ contains a uniformly convergent subsequence.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2007-2008

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 4 hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.