PhD Qualifying Examination

Algebra

Sem 1, 2007/2008

Answer all questions. Each question carries 25 marks.

(1) Let R be a Euclidean domain, and denote the $n \times n$-matrix ring over R by $M_n(R)$. Let $M \in M_n(R)$. Prove that there exist units $P, Q \in M_n(R)$ such that $PMQ = \operatorname{diag}(d_1, d_2, \dots, d_n)$ with $d_1 \mid d_2 \mid \cdots \mid d_n$.

(2) Prove that a simple group of order 60 is isomorphic to A_5.

(3) Let $K \subseteq L \subseteq M$ be fields. Prove or disprove each of the following statements:

(a) If L is algebraic over K and M is algebraic over L, then M is algebraic over K.
(b) If L is separable over K and M is separable over L, then M is separable over K.
(c) If L is Galois over K and M is Galois over L, then M is Galois over K.
(d) If L is radical over K and M is radical over L, then M is radical over K.

(4) Let R be a ring with multiplicative identity, and let M be a left R-module. Let $k \in \mathbb{Z}^+$. Prove that the following statements are equivalent:

(a) M is isomorphic to $R^k := \{(r_1, r_2, \dots, r_k) \mid r_i \in R \text{ for all } i\}$ as left R-modules.
(b) there exist $m_1, m_2, \dots, m_k \in M$ such that for every $m \in M$, there exist unique $r_1, \dots, r_k \in R$ such that $m = r_1 m_1 + r_2 m_2 + \cdots + r_k m_k$.
(c) there exist $m_1, m_2, \dots, m_k \in M$ such that every function f from $\{m_1, m_2, \dots, m_k\}$ to a left R-module N can be uniquely extended to a left module homomorphism $\tilde{f} : M \to N$.