NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2006-2007

MA3501 Mathematical Methods in Engineering

April/May 2007 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **SEVEN(7)** printed pages, including three appendices: Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.

3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Three Appendices (P5-P7): Formulae, Tables of Standard Normal Distribution and Boundary Value Problems.

Question 1 [14 marks]

(a) A research worker wants to determine the average time it takes a mechanic to rotate the tires of a car, and she wants to be able to assert with 95% confidence that the mean of her sample is off by at most 0.5 minute. If she can presume from past experience that the standard deviation is 1.6 minutes, how large a sample will she have to take?

(b) An instructor assigns grades in an examination according to the following procedure:

A if score exceeds $\mu + 1.5\sigma$;

B if score is between $\mu + 0.5\sigma$ and $\mu + 1.5\sigma$;

C if score is between $\mu - 0.5\sigma$ and $\mu + 0.5\sigma$;

D if score is between $\mu - 1.5\sigma$ and $\mu - 0.5\sigma$;

F if score is less than $\mu - 1.5\sigma$.

What percentage of each grade does this instructor give, assuming that the scores are normally distributed with mean μ and standard deviation σ?

Question 2 [14 marks]

Solve, by the method of characteristics, the following 1^{st} order PDE:

$$2u_t + e^t u_x = 2x,\ \ u(x,0) = 1 + x.$$

Question 3 [30 marks]

(a) Solve the following heat equation by the method of separation of variables:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \quad &&, \ 0 < x < 1,\ t > 0, \\ u_x(0,t) = 0,\ &u(1,t) = 0 \quad &&, \ t > 0, \\ u(x,0) &= x \quad &&, \ 0 < x < 1. \end{aligned}$$

(b) Solve the following heat equation:

$$\begin{aligned} \phi_t(x,t) &= \phi_{xx}(x,t) \quad &&, \ 0 < x < 1,\ t > 0, \\ \phi_x(0,t) = 1,\ &\phi(1,t) = 1 \quad &&, \ t > 0, \\ \phi(x,0) &= 0 \quad &&, \ 0 < x < 1. \end{aligned}$$

Question 4 [14 marks]

(a) Let c be a constant and ω a positive constant. Prove that the general solution of

$$y''(x) - \omega^2 y(x) = c$$

is $y(x) = Ae^{-\omega x} + Be^{\omega x} - \dfrac{c}{\omega^2}$.

(You can use any method to prove the above result.)

(b) Use the method of the Laplace transform (with respect to t) to find the solution of the following PDE:

$$\begin{aligned} u_{tt}(x,t) &= u_{xx}(x,t) + f(t) &&, \ x > 0,\ t > 0, \\ u(x,0) &= 0,\ u_t(x,0) = 0 &&, \ x > 0, \\ u(0,t) &= 0 &&, \ t > 0. \end{aligned}$$

(In solving the above PDE, we assume that $U(x,s)$ is bounded for $x > 0$, $s > 0$, where $U(x,s) = \mathcal{L}(u(x,t))$ = Laplace transform of $u(x,t)$.)

Question 5 [14 marks]

(a) Let γ_1 be a circle with centre 0 and radius 2 in anticlockwise direction.

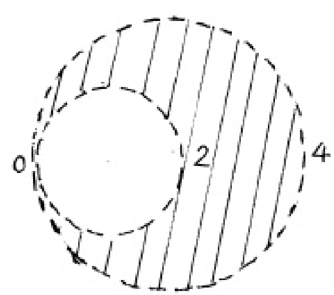
Evaluate the following integrals:

(i) $\displaystyle\int_{\gamma_1} \frac{z^2 + 3z + 2}{z^2(z-1)}\, dz$;

(ii) $\displaystyle\int_{\gamma_1} \frac{1}{z^2}\, dz$;

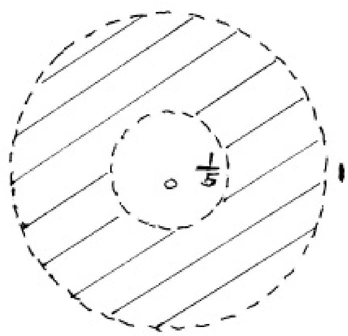
(iii) $\displaystyle\int_{\gamma_1} \frac{\sin(\frac{1}{z})}{z}\, dz$.

(b) Find the image of the crescent-shaped region, as shown in the figure below, that lies inside the open disk $|z-2| < 2$ and outside the circle $|z-1| = 1$ under the mapping $w = \dfrac{-iz + 4i}{z}$.



Question 6 [14 marks]

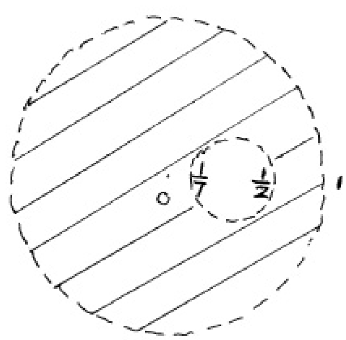
(a) Let $G = \{w : \frac{1}{5} < |w| < 1\}$, where $w = u + iv$.



Let $\Phi(w) = 50 + 50\dfrac{\ln|w| + \ln 5}{\ln 5}$. Prove that $\Phi(w)$ is a solution of Laplace's equation

$$\Phi_{uu}(w) + \Phi_{vv}(w) = 0 \text{ in } G.$$

(b) Let $D = \{z : |z < 1|\} \setminus \{z : |z - \frac{9}{28}| \leq \frac{5}{28}\}$.



(i) Solve the following PDE:

$$\begin{aligned} \phi_{xx}(z) + \phi_{yy}(z) = 0 &\quad \text{in } D, \\ \phi(z) = 50 &\quad \text{when} |z - \tfrac{9}{28}| = \tfrac{5}{28}, \\ \phi(z) = 100 &\quad \text{when} |z| = 1. \end{aligned}$$

Is the solution unique? Justify your answer.
(You may use the following fact without proof:
$w = \frac{3z-1}{3-z}$ maps D onto G and the boundary of D onto the boundary of G.)

(ii) Suppose $\phi(z)$ is the solution of PDE given in (i) of (b).Prove that $\phi(z) = \phi(\bar{z})$ and $50 < \phi(z) < 100$ for all z in D.

END OF PAPER

Formulae

(A) $\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)x'(t)$

(B) $\int_0^1 x\cos\left[\frac{(2n-1)\pi x}{2}\right]\,dx = \frac{(-1)^{n+1}\cdot 2}{(2n-1)\pi} - \frac{4}{(2n-1)^2\pi^2}, n = 1, 2, \ldots$

(C) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$

(D) $\text{Log} z = \ln|z| + i\text{Arg} z,\ -\pi < \text{Argz} < \pi.$

(E) **LaplaceTransforms**

Laplace transform with respect to t denoted by $\mathcal{L}$.

$\mathcal{L}(u(x,t))$ denoted by $U(x,s)$.

$\mathcal{L}(u_{xx}(x,t)) = \frac{d^2}{dx^2}U(x,s)$.

$\mathcal{L}(u_{tt}(x,t)) = s^2U(x,s) - su(x,0) - u_t(x,0)$.

$\mathcal{L}(f(t))$ denoted by $F(s)$.

$\mathcal{L}([t-(t-x)H(t-x)]) = \frac{1-e^{-sx}}{s^2}$, where $H(w) = \begin{cases} 0 & ;\text{ if } w < 0 \\ 1 & ;\text{ if } w > 0 \end{cases}$.

$\mathcal{L}((f*g)(t)) = \mathcal{L}(f(t))\mathcal{L}(g(t))$.

$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$.