2007 Exam solns

Q1

(a) $E = 0.50 \quad \sigma = 1.6 \quad Z_{0.025} = 1.96$

$$n = \left[\frac{Z_{0.025}\ \sigma}{E}\right]^2$$

$$= \left(\frac{(1.96)(1.6)}{0.5}\right)^2$$

$$= 39.3$$

Take $n = 40$

(b)

A iff $z \geq 1.5$ $\therefore$ Prob $= 0.0668$, 6.68%

B iff $0.5 \leq z \leq 1.5$ $\therefore$ Prob $= 0.2417$, 24.17%

C iff $-0.5 \leq z \leq 0.5$ $\therefore$ Prob $= 0.3830$, 38.30%

D iff $-1.5 \leq z \leq -0.5$ - - - - . 24.17%

E iff $z \leq -1.5$ - - - - - - . 6.68%

100%

Q2

$$u_t + \frac{1}{2} e^t u_x = x$$

$$u(x,0) = 1+x .$$

Ans : $u(x,t) = 1 + x + xt - \frac{1}{2} t e^x$

Q3

(a) From the 5th column of BVP table

we get

$$\lambda_n = \frac{(2n-1)^2 \pi^2}{4}$$

$$X_n(x) = \cos \frac{(2n-1)\pi x}{2}$$

$$n = 1, 2, \ldots$$

Ans : $u(x,t) = \sum_{n=1}^{\infty} \left[(-1)^{n+1} \frac{4}{(2n-1)\pi} - \frac{8}{(2n-1)^2 \pi^2} \right] \cos \frac{(2n-1)\pi x}{2} e^{-\frac{(2n-1)^2 \pi^2 t}{4}}$

(b) let $\phi(x,t) \longrightarrow V(x)$ as $t \to \infty$

then we have

$$\left. \begin{array}{l} v''(x) = 0 \\ v'(0) = 1 \\ v(1) = 1 \end{array} \right\} \Rightarrow V(x) = x$$

let $w = \phi - V$. then
$$\begin{array}{l} w_t = w_{xx} \\ w_x(0,t) = 0 \\ w(1,t) = 0 \\ w(x,0) = -x \end{array}$$

$\therefore$ $\phi = w + V = -u + V = -u + x$

where u given in (a)

Q4

(a) $y = Ae^{-wx} + Be^{wx} - \frac{c}{w^2}$

$y' = -wAe^{-wx} + Bwe^{wx}$

$y'' = w^2Ae^{-wx} + Bw^2e^{wx}$

$y'' - w^2y = \cdots\cdots = c$

(b)

let $\mathcal{L}u(x,t) = U(x,s)$, $\mathcal{L}f(t) = F(s)$

$s^2U(x,s) - su(x,0) - \frac{du}{dt}(x,0) = \frac{d^2U(x,s)}{dx^2} + F(s)$

$\therefore \cancel{\frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = F(s)}$

$\frac{d^2U(x,s)}{dx^2} - s^2U(x,s) = -F(s)$

From (a) $U(x,s) = A(s)e^{-sx} + B(s)e^{sx} + \frac{F(s)}{s^2}$

$B(s) = 0$, $A(s) = -\frac{F(s)}{s^2}$

$$U(x,s) = F(s)\frac{1-e^{-\frac{s}{c}x}}{s^2}$$

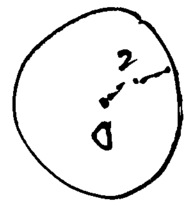
$$u(x,t) = f(t) * \mathcal{L}^{-1}\left(\frac{1-e^{-\frac{s}{c}x}}{s^2}\right)$$

$$= f(t) * \left[t - \left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)\right]$$

$$u(x,t) = \int_0^t f(t-\tau)\left[\tau - \left(\tau-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)\right]d\tau$$

Q(5)

(a)



(i) $\int_{\gamma_1} \frac{z^2+3z+2}{z^2(z-1)} dz$

$= 2\pi i \left(\text{Res}(0) + \text{Res}(1)\right)$

$= 2\pi i \left(-5+6\right)$

(ii) $\int_{\gamma_1} \frac{1}{z^2} dz = 2\pi i \text{ Res}(0) = 2\pi i (0)$

(iii) $\int_{\gamma_1} \frac{\sin(\frac{1}{z})}{z} = 2\pi i \text{ Res}(0) = 2\pi i (0) = 0$

$\text{Res}(0) = 0$

$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{z^3}\frac{1}{3!} + \cdots$

$\frac{\sin\frac{1}{z}}{z} = \frac{1}{z^2} - \frac{1}{z^4}\frac{1}{3!} + \cdots$

Q15)

(b) Choose three pts

$z_1 = 4, \; z_2 = 2+2i, \; z_3 = 0$

on the boundary of $|z-2| < 2$

The images are

$w_1 = 0, \; w_2 = 1, \; w_3 = \infty$ resp.

$\therefore$ boundary of $|z-2| < 2$ maps to u-axis ($\text{Im}(w) = 0$)

Choose three pts on the boundary

of $|z-1| < 1$

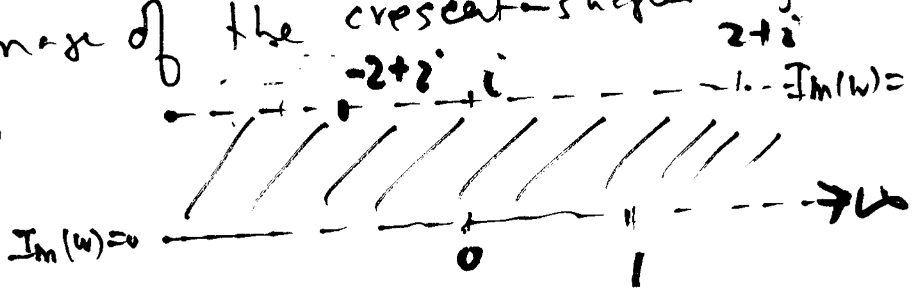
$z_4 = 1-i, \; z_5 = 2, \; z_6 = 1+i$

The images are

$w_4 = -2+i, \; w_5 = i, \; w_6 = 2+i$ resp

$\therefore$ boundary of $|z-1| < 1$ maps to the horizontal line $\text{Im}(w) = 1$

Image of the crescent-shaped region is



Q6

(a)

$\ln|w|$ is the real part of $\mathrm{Log}\, w$.

$\therefore$ $\Phi(w) = 50 + 50 \dfrac{\ln|w| + \ln 5}{\ln 5}$

is a soln of Laplace's eq in G

Note: if $u(w)$ is a soln of L. eq. then

(1)

$a\,u(w) + b$ is a soln of L. eq.

(b)

(i) $\Phi(w) = 50 + 50 \dfrac{\ln|\frac{1}{5}| + \ln 5}{\ln 5}$

$= 50 + 50 \dfrac{0}{\ln 5} = 50$

if w is on the boundary

of inner circle of G.

$\Phi(w) = 50 + 50 \dfrac{\ln 1 + \ln 5}{\ln 5}$

$= 100$

if w is on the boundary of

outer circle of G.

$\phi(z) = \Phi(f(z)) = 50 + 50 \dfrac{\ln\left|\frac{3z-1}{3-z}\right| + \ln 5}{\ln 5}$

The soln is unique by maxi-mini temperature principle.

(Suppose two solns say ϕ_1, ϕ_2.

Let $w = \phi_1 - \phi_2$, then $w = 0$ on boundary.

By maxi-mini principle $w = 0$ everywhere

$\therefore \phi_1 = \phi_2$)

(iii) ~~$\Phi(w)$~~ $\ln\left|\frac{3z-1}{z-2}\right| = \ln\left|\frac{3\bar{z}-1}{\bar{z}-2}\right|$

$$\left|\frac{3\bar{z}-1}{\bar{z}-2}\right| = \left|\overline{\left(\frac{3z-1}{z-2}\right)}\right| = \left|\frac{3z-1}{z-2}\right|$$

$\therefore \phi(z) = \phi(\bar{z})$

Or temperature dist. is symmetric w.r.t. to x-axis.

$50 < \phi(z) < 100$ for all z in D

by maxi-mini temperature principle