NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 2 EXAMINATION 2006-2007

MA1506 MATHEMATICS II

April 2007 Time allowed: 2 hours

Matriculation Number:

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation number neatly in the space provided above.** This booklet (and only this booklet) will be collected at the end of the examination. Do not insert any loose pages in the booklet.
2. This examination paper consists of **EIGHT (8)** questions and comprises **THIRTY THREE (33)** printed pages.
3. Answer **ALL** questions. For each question, write your answer in the box and your working in the space provided inside the booklet following that question.
4. The marks for each question are indicated at the beginning of the question.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

For official use only. Do not write below this line.

Question	1	2	3	4	5	6	7	8
Marks								

Question 1 (a) [5 marks]

(i) Find the derivative of the function $ln(ln(x))$ defined on the domain $x > 1$.
(ii) If $y(x)$ satisfies the differential equation

$$x(\ln x)\frac{dy}{dx} + y - x^2 = 0, \qquad x > 1$$

and $y(e) = 0$, find $y(e^2)$.

Answer 1(a)(i)		**Answer 1(a)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 1(a))

Question 1 (b) [5 marks]

The number of neutrons N at time t flying around inside a nuclear reactor satisfies the differential equation

$$\frac{dN}{dt} = 0.01N^2 - N.$$

At time $t = 0$ second, there are 101 neutrons. Find the number of neutrons in the reactor at time $t = 4.61$ seconds.

Answer 1(b)	

(Show your working below and on the next page.)

(More working space for Question 1(b))

Question 2 (a) [5 marks]

Solve the differential equation

$$y'' + 2y' + y = xe^{-x}$$

with the initial conditions that $y = 1$ and $y' = 0$ when $x = 0$.

Answer 2(a)	

(Show your working below and on the next page.)

(More working space for Question 2(a))

Question 2 (b) [5 marks]

Solve the differential equation

$$y'' + y = \sec x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

with the initial conditions that $y = 1$ and $y' = 1$ when $x = 0$.

Answer 2(b)	

(Show your working below and on the next page.)

(More working space for Question 2(b))

Question 3 (a) [5 marks]

(i) Find the stable equilibrium point of the differential equation

$$\ddot{x} = \cos(\pi \sin x)$$

in the domain $0 < x < \frac{\pi}{2}$.

(ii) Find the approximate angular frequency of oscillation around this equilibrium point.

Answer 3(a)(i)		**Answer 3(a)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 3(a))

Question 3 (b) [5 marks]

On the island of Progensia, the human birth rate per capita B and the human death rate per capita D are constants and the population of the island doubles every 15 years. However, one day several pirate ships arrive. All of the island women under the age of 50, tired of being ordered about by their mothers-in-law and ignored by their husbands, decide to elope with the glamorous pirates, taking their children with them. After that, the human birth rate per capita $\widetilde{B}$ and the human death rate per capita $\widetilde{D}$ stay constant with $\widetilde{B} = \frac{1}{20}B$ and $\widetilde{D} = \frac{3}{2}D$, and the remaining population of Progensia declines by half over the next ten years. What was the original birth rate per capita on Progensia?

Answer 3(b)	

(Show your working below and on the next page.)

(More working space for Question 3(b))

Question 4 (a) [5 marks]

The elephant bird is a large bird in Madagascar with an unfortunate liking for farm crops. For many years the birds were protected by law, and eventually they settled down to a logistic equilibrium population of 200000 with birth rate 10% per year. Eventually the patience of the farmers was exhausted and they managed to persuade their government to allow them to shoot 3000 elephant birds per year. What is the limiting population in this case?

Answer 4(a)	

(Show your working below and on the next page.)

(More working space for Question 4(a))

Question 4 (b) [5 marks]

Biologists use the differential equation

$$\frac{dN}{dt} = N\left[0.1 - 0.001\left(N - 100\right)^2\right]$$

to model a certain rare species of tigers which has a current population equal to $\widehat{N}$. Let A be the largest number such that if $\widehat{N} < A$, then this tiger population will become extinct.

(i) Find the value of A.

(ii) If $\widehat{N} = 1000$ what is the limiting population?

Answer 4(b)(i)		**Answer 4(b)(ii)**	

(Show your working below and on the next page.)

(More working space for Question 4(b))

Question 5 (a) [5 marks]

Solve the initial-value problem

$$y'' + y = 0.5\delta(t - 2\pi),$$
$$y(0) = 1, \quad y'(0) = 0.$$

Answer 5(a)	

(Show your working below and on the next page.)

(More working space for Question 5(a))

Question 5 (b) [5 marks]

Find the value of the inverse Laplace transform of

$$\frac{e^{-\pi s}}{s^2 + 2s + 5}$$

at $t = 5\pi/4$.

Answer 5(b)	

(Show your working below and on the next page.)

(More working space for Question 5(b))

Question 6 (a) [5 marks]

The exponential of a square matrix B is defined to be the infinite series

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + ...,$$

where I denotes the identity matrix. Find the exponential of the matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Answer 6(a)	

(Show your working below and on the next page.)

(More working space for Question 6(a))

Question 6 (b) [5 marks]

The weather in a certain city can be rainy, foggy, or sunny. If it is rainy today, the probability of rain tomorrow is 0.3 and the probability that it will be foggy is 0.4. Similarly other probabilities are foggy $\rightarrow$ rain $= 0.7$, foggy $\rightarrow$ foggy $= 0.2$, sunny $\rightarrow$ rain $= 0.5$ and sunny $\rightarrow$ foggy $= 0.2$. Given that it is foggy today, what is the probability of fog three days from now?

Answer 6(b)	

(Show your working below and on the next page.)

(More working space for Question 6(b))

Question 7 (a) [5 marks]

In two dimensions, we perform a shear [shearing angle 45 degrees] by means of forces parallel to an axis which makes an angle of 30 degrees with respect to the positive x-axis. Find the final location of the point which was originally at (x, y) coordinates (0, 1).

Answer 7(a)	

(Show your working below and on the next page.)

(More working space for Question 7(a))

Question 7 (b) [5 marks]

By diagonalizing the matrix $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$, compute its fourth power. [**Zero marks if you don't diagonalize.**]

Answer 7(b)	

(Show your working below and on the next page.)

(More working space for Question 7(b))

Question 8 (a) [5 marks]

Classify the phase plane diagrams of the systems of linear differential equations having the following matrices: $\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$, $\begin{pmatrix} 2 & -2 \\ 8 & 0 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $\begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$.

Answer 8(a)	

(Show your working below and on the next page.)

(More working space for Question 8(a))

Question 8 (b) [5 marks]

King Xerxes has sent an army of 50000 Persian warriors to attack the Spartans at Thermopylae. Each Persian can kill enemy soldiers at an average rate of 1/2 per unit of time, but the Spartans can kill an average of 15/2 enemies in that time. Furthermore, the Persian army suffers from a disease which kills them, according to a Malthus model, with a death rate per capita of 1 per unit of time. What is the minimum number of soldiers the Spartans need in order to destroy the entire Persian army?

Answer 8(b)	

(Show your working below and on the next page.)

(More working space for Question 8(b))

END OF PAPER

Question 1(a)(i)

$$\frac{d}{dx}\ln(\ln x) = \frac{1}{\ln x}\left(\frac{d}{dx}\ln x\right) = \frac{1}{x\ln x}.$$

Question 1(a)(ii)

$$\frac{dy}{dx} + \frac{1}{x\ln x}y = \frac{x}{\ln x}.$$

Integrating factor $= e^{\int \frac{1}{x\ln x}} = e^{\ln\ln x} = \ln x$. Hence,

$$\begin{aligned} y\ln x &= \int \frac{x}{\ln x}\ln x = \frac{x^2}{2} + C \\ \implies y(e)\ln e = 0 &= \frac{e^2}{2} + C \implies C = -\frac{e^2}{2}. \end{aligned}$$

$$\begin{aligned} y\ln x &= \frac{x^2}{2} - \frac{e^2}{2} \\ \implies y(e^2)\ln e^2 &= \frac{e^4}{2} - \frac{e^2}{2} \\ \implies y(e^2) &= \frac{e^4 - e^2}{4}. \end{aligned}$$

Question 1(b)

$$\frac{dN}{N(N-100)} = 0.01dt \implies \left(\frac{1}{N-100} - \frac{1}{N}\right) dN = dt$$
$$\implies \ln|N-100| - \ln|N| = t + C$$
$$\implies \frac{N-100}{N} = Ae^t.$$

$N(0) = 101 \implies A = 1/101$. Therefore,

$$\frac{N-100}{N} = \frac{1}{101}e^t \implies N = \frac{10100}{101-e^t}$$
$$\implies N(4.61) = \frac{10100}{101-e^{4.61}} \approx 19579.$$

Question 2(a)

For homogeneous solution y_h, solve

$$\lambda^2 + 2\lambda + 1 = 0 \implies \lambda = -1.$$

Hence $y_h = (Ax + B)e^{-x}$. For y_p, try

$$\begin{aligned} y = ue^{-x} &\implies y' = u'e^{-x} - ue^{-x} \\ &\implies y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}. \end{aligned}$$

So

$$\begin{aligned} &u''e^{-x} - 2u'e^{-x} + ue^{-x} + 2\left(u'e^{-x} - ue^{-x}\right) + ue^{-x} = xe^{-x} \\ &\implies u'' = x \implies u = \frac{1}{6}x^3 + Cx + D. \end{aligned}$$

Thus $y = (Ax + B)e^{-x} + \frac{1}{6}x^3e^{-x}$.
$y(0) = 1 \implies B = 1$.

Since

$$y' = Ae^{-x} - (Ax + 1)e^{-x} + \frac{1}{2}x^2e^{-x} - \frac{1}{6}x^3e^{-x}.$$

$y'(0) = 0 \implies A - 1 = 0.$

Finally,

$$y = (x + 1)e^{-x} + \frac{1}{6}x^3e^{-x} = (1 + x + \frac{1}{6}x^3)e^{-x}.$$

Question 2(b)

For homogeneous solution y_h, solve

$$\lambda^2 + 1 = 0 \implies \lambda = \pm i.$$

Hence $y_h = A\cos x + B\sin x$.

Wronskian $W = \cos^2 x - (-\sin^2 x) = 1$, so

$$\begin{aligned} u &= -\int \sin x \sec x = -\int \frac{\sin x}{\cos x} = \ln(\cos x) \\ v &= \int \cos x \sec x = \int 1 = x. \end{aligned}$$

Hence $y = A\cos x + B\sin x + \ln(\cos x)\cos x + x\sin x$.
$y(0) = 1 \implies A = 1$.

$y' = -\sin x + B\cos x - \ln(\cos x)\sin x - \tan x \cos x + \sin x + x\cos x$.
$y'(0) = 1 \implies 1 = B$. Hence

$$y = \cos x + \sin x + \ln(\cos x)\cos x + x\sin x.$$

Question 3(a)(i)

$$0 < x < \frac{\pi}{2} \implies 0 < \sin x < 1 \implies 0 < \pi \sin x < \pi.$$

$$\text{hence} \quad \cos(\pi \sin x) = 0 \implies \pi \sin x = \frac{\pi}{2} \implies x = \frac{\pi}{6}.$$

Using Taylor's theorem,

$$\begin{aligned}
\ddot{x} &\approx \frac{d}{dx} \cos(\pi \sin x)\bigg|_{x=\frac{\pi}{6}} \left(x - \frac{\pi}{6}\right) \\
&= -\sin(\pi \sin x)\pi \cos x|_{x=\frac{\pi}{6}} \left(x - \frac{\pi}{6}\right) \\
&= -\sin(\pi(\frac{1}{2}))\pi \cos(\frac{\pi}{6}) \left(x - \frac{\pi}{6}\right) \\
&= -\frac{\sqrt{3}}{2}\pi \left(x - \frac{\pi}{6}\right).
\end{aligned}$$

Set

$$y = x - \frac{\pi}{6} \implies \ddot{y} \approx -\frac{\sqrt{3}}{2}\pi y.$$

Hence $x = \frac{\pi}{6}$ is a stable equilibrium.

Question 3(a)(ii)

$$\omega \approx \sqrt{\frac{\sqrt{3}}{2}\pi} \approx 1.65.$$

Question 3(b)

Before the pirates,

$$\begin{aligned} N &= \hat{N}e^{(B-D)t} \\ 2\hat{N} &= \hat{N}e^{(B-D)15} \implies B - D = \frac{1}{15}\ln 2. \end{aligned}$$

After the pirates,

$$\begin{aligned} N = \tilde{N}e^{(\tilde{B}-\tilde{D})t} & \\ \frac{1}{2}\tilde{N} = \tilde{N}e^{(\tilde{B}-\tilde{D})10} &\implies \tilde{B} - \tilde{D} = -\frac{1}{10}\ln 2 \\ &\implies 0.05B - 1.5D = -\frac{1}{10}\ln 2. \end{aligned}$$

Solving for B ,

$$1.5B - 0.05B = 2(\frac{1}{10}\ln 2) \implies B \approx 0.0956.$$

or $B \approx 9.56\%$.

Question 4(a)

$$B = 0.1, N_\infty = 200,000 \implies s = 0.1/200000 = 5 \times 10^{-7}.$$

Hence,

$$\begin{aligned}\frac{dN}{dt} &= BN - sN^2 - E \\ &= 0.1N - 5 \times 10^{-7}N^2 - 3000.\end{aligned}$$

Since $B^2/4s = BN_\infty/4 = 5000 > E$, we have two equilibriums given by the roots of $BN - sN^2 - E$. Solving, this quadratic gives us two roots, $\beta_1 \approx 36754$ (unstable) and $\beta_2 \approx 163246$ (stable).

Limiting population is then ≈ 163246.

Question 4(b)

$$\begin{aligned}\frac{dN}{dt} &= 0.001N(100 - (N - 100)^2) \\ &= -0.001N(N - 90)(N - 110).\end{aligned}$$

The curve $F(N) = \frac{dN}{dt}$ has 3 roots, giving us 3 equilibriums. $N = 90$ is an unstable equilibrium while $N = 110$ is stable.

(i) $A = 90$
(ii) If $\hat{N} > 110$ then $N \to 110$.

Question 5(a)

Apply Laplace transform to get,

$$\begin{aligned} L(y'') &= s^2Y - sy(0) - y'(0) = s^2Y - s \\ L(y) &= Y \\ L(0.5\delta(t-2\pi)) &= 0.5e^{-2\pi s}. \end{aligned}$$

Hence

$$(s^2+1)Y = s + 0.5e^{-2\pi s} \implies Y = \frac{s}{s^2+1} + \frac{0.5e^{-2\pi s}}{s^2+1}.$$

Since $L(f(t-a)u(t-a)) = e^{-as}F(s)$ and $L(sint) = \frac{1}{s^2+1}$,

$$y = L^{-1}(Y) = \cos t + \frac{1}{2}\sin(t-2\pi)u(t-2\pi).$$

Question 5(b)

Completing the square,

$$\frac{e^{-\pi s}}{s^2+2s+5} = \frac{e^{-\pi s}}{(s+1)^2+2^2}.$$

Using s-shifting,

$$L(e^{-t}\sin 2t) = \frac{2}{(s+1)^2+2^2}.$$

Using t-shifting, $L(f(t-a)u(t-a)) = e^{-as}F(s)$, we have

$$L^{-1}\left(\frac{e^{-\pi s}}{s^2+2s+5}\right) = \frac{1}{2}e^{-(t-\pi)}\sin(2(t-\pi))u(t-\pi)$$

$$L^{-1}\left(\frac{e^{-\pi s}}{s^2+2s+5}\right)\Bigg|_{t=\frac{5\pi}{4}} = \frac{1}{2}e^{-\frac{\pi}{4}}.$$

Question 6(a)

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad A^3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\begin{aligned} e^A &= I + A + \frac{1}{2}A^2 \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & \frac{3}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Question 6(b)

$$M = \begin{pmatrix} R\to R & F\to R & S\to R \\ R\to F & F\to F & S\to F \\ R\to S & F\to S & S\to S \end{pmatrix} = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix}.$$

$$\begin{aligned} M^2 &= \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix} \\ M^3 &= \begin{pmatrix} 0.52 & 0.4 & 0.44 \\ 0.26 & 0.34 & 0.3 \\ 0.22 & 0.26 & 0.26 \end{pmatrix} \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.2 & 0.2 \\ 0.3 & 0.1 & 0.3 \end{pmatrix} \\ &= \begin{pmatrix} 0.448 & 0.448 & 0.472 \\ 0.304 & 0.28 & 0.288 \\ 0.248 & 0.232 & 0.24 \end{pmatrix} \end{aligned}$$

Answer = 0.28.

Question 7(a)

Rotate -30°, shear 45° parallel to x-axis, then rotate 30°.

$$
\begin{aligned}
T &= \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \begin{pmatrix} 1 & \tan 45 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos 30 & \sin 30 \\ -\sin 30 & \cos 30 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} - \frac{1}{2} & \frac{\sqrt{3}}{2} + \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.
\end{aligned}
$$

$$
T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & 1 + \frac{\sqrt{3}}{4} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{4} \\ 1 + \frac{\sqrt{3}}{4} \end{pmatrix}.
$$

Answer $= (\frac{3}{4}, 1 + \frac{\sqrt{3}}{4})$.

Question 7(b)

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$
$$\implies \lambda^2+\lambda-6=0$$
$$\implies \lambda = -3, 2.$$

$\lambda = -3$,

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \alpha = -2.$$

$\lambda = 2$,

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \beta = \frac{1}{2}.$$

So

$$P = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix} \implies P^{-1} = \frac{2}{5}\begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix}.$$

Hence

$$\begin{aligned} A^4 &= \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} (-3)^4 & 0 \\ 0 & 2^4 \end{pmatrix}\frac{2}{5}\begin{pmatrix} \frac{1}{2} & -1 \\ 2 & 1 \end{pmatrix} \\ &= \frac{2}{5}\begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{81}{2} & -81 \\ 32 & 16 \end{pmatrix} \\ &= \begin{pmatrix} 29 & -26 \\ -26 & 68 \end{pmatrix}. \end{aligned}$$

Question 8(a)

$$\begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$$

Trace $= -1$, Det $= -4 \implies$ Saddle.

$$\begin{pmatrix} 2 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace $= 2$, Det $= 16 > 0$
Tr^2 - 4Det $= -60 < 0 \implies$ Spiral Source.

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Trace $= 4$, Det $= 3 > 0$
Tr^2 - 4Det $= 4 > 0 \implies$ Nodal Source.

$$\begin{pmatrix} 0 & -2 \\ 8 & 0 \end{pmatrix}$$

Trace $= 0$, Det $= 16 > 0$
Tr^2 - 4Det $= -64 < 0 \implies$ Centre.

Question 8(b)

Let P, S be the number of Persians and Spartans respectively.

$$\begin{aligned} \frac{dP}{dt} &= -P - \tfrac{15}{2}S \\ \frac{dP}{dt} &= -\tfrac{1}{2}P + 0 \end{aligned} \implies \begin{pmatrix} -1 & -\frac{15}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} P \\ S \end{pmatrix}.$$

Since det < 0,we have a saddle.

$$\begin{vmatrix} -1-\lambda & -\frac{15}{2} \\ -\frac{1}{2} & -\lambda \end{vmatrix} = 0 \implies \lambda^2 + \lambda - \frac{15}{4} = 0$$

$$\implies \lambda = -\frac{5}{2}, \frac{3}{2}.$$

The corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ \frac{1}{5} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{3} \end{pmatrix}.$$

Using the first eigenvector, we see that as long as $S > 10000$, the solution curve will intersect the vertical axis where $P = 0$.