NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER II EXAMINATION 2006-2007

MA5213 Advanced Partial Differential Equations

April 27, 2007 — Time allowed : 2 and 1/2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper consists of **TWO (2)** sections: Section A and Section B. It contains a total of **SEVEN (7)** questions and comprises **Three (3)** printed pages.
2. Answer **ALL** questions in **Section A** . The marks for questions in Section A may not necessarily be the same; marks for each question are indicated at the beginning of the question.
3. Answer not more than **TWO** questions from **Section B**. Each question in Section B carries 20 marks.
4. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

SECTION A

Answer **all** *the questions in this section. Section A carries a total of 60 marks.*

Question 1 [10 marks]

Show that a function u is weakly differentiable in a domain Ω if and only if it is weakly differentiable in a neighborhood of every point in Ω.

Question 2 [20 marks]

Let Ω be a bounded domain in R^n with C^1 boundary. Assume that $u \in H^1(\Omega)$ and there exist constants $K > 0$ and $0 < \alpha < 1$ such that

$$\int_{B_R} |Du| dx \leq K R^{n-1+\alpha}$$

for all $B_R \subset \Omega$. Show that $u \in C^{0,\alpha}(\Omega)$ and for all ball $B_R \subset \Omega$,

$$\max_{B_R} u - \min_{B_R} u \leq C K R^\alpha,$$

for some constant $C = C(n, \alpha)$.

Question 3 [10 marks]

Let f be a measurable function on a domain in R^n. The distribution function $\mu = \mu_f$ is defined by

$$\mu(t) = \mu_f(t) = | \{x \in \Omega : \ |f|(x) > t\} |$$

for $t > 0$. Show that if $p > 0$ and $|f|^p \in L^1(\Omega)$, then

$$\mu(t) \leq t^{-p} \int_\Omega |f|^p dx.$$

Question 4 [20 marks]

If Ω is a convex set in R^n, show that, for any measurable subset $S \subset \Omega$ and $x \in \Omega$, $|u(x) - u_S| \leq \frac{d^n}{n|S|} V_{\frac{1}{n}}(|Du|)(x)$ for all $u \in W^{1,p}$ with $p \geq 1$ where d is the diameter of Ω, $V_{\frac{1}{n}}(|Du|)$ is the Riesz-potential of $|Du|$ and u_S stands for the average value of u over S. Use this or otherwise to conclude that

$$\|u - u_\Omega\|_{L^p(\Omega)} \leq n\left(\frac{\omega_n}{|\Omega|}\right)^{1-(1/n)} d^n \|Du\|_{L^p(\Omega)},$$

where ω_n is the volume of unit ball $B_1(0)$ in R^n.

SECTION B

Answer not more than **two** *questions from this section. Each question in this section carries 20 marks.*

Question 5 [20 marks]

Let Ω be a smooth domain in R^n and $u \in W^{2,n}(\Omega)$ satisfy the equation $\Delta u \leq f$ in Ω, where $f \in L^n(\Omega)$. Suppose that u is non-negative in a ball $B_R(y) \subset \Omega$. Show that there are constants $p > 0$ and $C > 0$ such that

$$(\frac{1}{|B_R|}\int_{B_R} u^p dx)^{1/p} \leq C(\inf_{B_R} u + R\|f\|_{L^n(\Omega)}),$$

where constants C and p depend only on n.

Question 6 [20 marks]

By using variational method, show that the boundary value problem $u'' + u^p = 0$ in the interval $(0,1)$ with $u(0) = u(1) = 0$ has a positive classical solution where $p > 1$ is a real number.

Question 7 [20 marks]

(a) Let X be a Banach space and T be a bounded linear mapping of X into itself satisfying

$$\|x\| \leq K\|Tx\|$$

for all $x \in X$ and for some $K \in R^+$. Show that the range of T is closed.

(b) Prove that a bounded sequence in a reflexive, separable Banach space contains a weakly convergent subsequence.

END OF PAPER