NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER I EXAMINATION 2006-2007

MA5205 Graduate Analysis I

November/December 2006— Time allowed : 2 and 1/2 hours

INSTRUCTIONS TO CANDIDATES

(1) This examination paper consists of **TWO** (2) sections: Section A and Section B. It contains a total of **SIX** (6) questions and comprises **FOUR** (4) printed pages.

(2) Answer **ALL** questions in **Section A**. The marks for questions in Section A are not necessarily the same; marks for each question are indicated at the beginning of the question.

(3) Answer not more than **TWO** (2) questions from **Section B**. Each question in Section B carries 20 marks.

(4) Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

(5) All theorems and results used in your answers should be clearly stated.

SECTION A

Answer **all** the questions in this section. Section A carries a total of 60 marks.

Question 1 [36 marks]

Prove or disprove each of the following statements.

(i) Let f be a Lebesgue integrable function on $\mathbb{R}^n$. If $\{f_n\}$ and $\{g_n\}$ are sequences of simple functions that converge to f almost everywhere, then $\lim_{n\to\infty}\int f_n = \lim_{n\to\infty}\int g_n$. Note that a function is simple if and only if it is of the form

$$\sum_{i=1}^{N} a_i \chi_{E_i} \quad \text{with } |E_i| < \infty \quad \text{for all } i.$$

(ii) A real function is measurable if $\{x \in \mathbb{R} : f(x) = \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

(iii) If f is a function of bounded variation on $[a,b] \subset \mathbb{R}$, then it is also continuous on $[a,b]$.

(iv) If f is a Lebesgue measurable function on $\mathbb{R}$, then it is continuous almost everywhere on $\mathbb{R}$.

(v) If $F, K \subset \mathbb{R}^n$ such that F is closed and K is compact, then

$$\inf_{x\in F, y\in K} |x-y| > 0$$

if and only if $F \cap K = \emptyset$.

(vi) If $D^- f(x) = \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} \geq 0$ for almost all $x \in [a,b]$ and f is an absolutely continuous function on $[a,b]$, then f is nondecreasing.

Question 2 [18 marks]

(a) If f is a finite valued Lebesgue measurable function on $\mathbb{R}^n$, show that there exists a Borel measurable function h such that $f = h$ almost everywhere.

(b) If there exists $C > 0$ such that

$$|\{x \in \mathbb{R}^n : |f(x)| > t\}| \leq C \min\{t^{-1}, t^{-q}\} \text{ for all } t > 0,$$

show that $f \in L^p(\mathbb{R}^n)$ for all $1 < p < q$.

Question 3 [6 marks]

If $f \in L^p(\mathbb{R}^n)$ for some $p > 1$, show that given any $\varepsilon > 0$, there exists a compact set $K \subset \mathbb{R}^n$ such that $\int_{\mathbb{R}^n \setminus K} |f|^p < \varepsilon$.

SECTION B

Answer not more than **two** questions in this section. Each question in this section carries 20 marks.

Question 4 [20 marks]

Let w be a measurable function on $\mathbb{R}$ such that $w > 0$ almost everywhere on $\mathbb{R}$.

(i) Show that $L^2_w(\mathbb{R})$ is a Hilbert space, where

$$f \in L^2_w(\mathbb{R}) \text{ if } \int f(x)^2 w(x)dx < \infty.$$

(ii) If $f \in L^1(\mathbb{R})$ such that $f^* \in L^2_w(\mathbb{R})$ (where f^* is the Hardy-Littlewood maximal function of f) and

$$K_y(x) = \frac{1}{\pi}\frac{y}{|x|^2 + y^2}, \quad \text{for } x \in \mathbb{R}, y > 0 \ ,$$

show that $f * K_y \to f$ in $L^2_w(\mathbb{R})$ as $y \to 0^+$.

Question 5 [20 marks]

Let w be a measurable function on $\mathbb{R}^n$ such that $w > 0$ almost everywhere on $\mathbb{R}^n$ and

$$\int_{2B} w(x)dx \leq \lambda \int_B w(x)dx < \infty$$

for any open ball $B \subset \mathbb{R}^n$ where $2B$ is the ball concentric with B with twice the radius of B and $\lambda > 0$ is a fixed constant. For any measurable set $E \subset \mathbb{R}^n$, we define $w(E) = \displaystyle\int_E w$. For any measurable function f on $\mathbb{R}^n$, define

$$f^\sharp(x) = \sup\{\frac{1}{w(B)}\int_B |f|wdy : B \text{ is an open ball that contains } x\}.$$

If $\int |f|wdx < \infty$, show that there exists constant $C > 0$ such that

$$w\{x \in \mathbb{R}^n : f^\sharp(x) > \alpha\} \leq C/\alpha \quad \text{for all } \alpha > 0.$$

Question 6 [20 marks]

(a) Let $v \in L^1[a, b]$. If g is an absolutely continuous function on $[a, b]$, such that $\int_a^b gv = 0$, show that

$$g(x)v[a, b] = \int_a^x v[a, t]g'(t)dt - \int_x^b v[t, b]g'(t)dt \quad \text{for all } x \in [a, b]$$

where $v[c, d] = \int_c^d v$.

(b) Let $\{f_n\}$ be a sequence of measurable functions that converges to a function f almost everywhere. Let $\{g_n\}$ be a sequence of integrable functions that converges to an integrable function g almost everywhere. If $|f_n| \leq g_n$ almost everywhere for all n and

$$\lim_{n\to\infty} \int g_n = \int g,$$

show that

$$\lim_{n\to\infty} \int |f_n - f| = 0, \quad \text{and} \quad \lim_{n\to\infty} \int f_n = \int f.$$

END OF PAPER