NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2006-2007

MA5203 Graduate Algebra I

November 2006 – Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer not more than **FOUR (4)** questions.
3. Maximum marks will be allocated as follows:

Your best 3 answers	:	27% each	81%
Next best answer	:	19%	19%

			100%

4. Results *proved* in lectures or tutorial assignments that you use should be stated clearly but need not be proved.
5. The symbol R always refers to a ring (with 1), assumed arbitrary, unless otherwise stated.

Question 1. Let M be an abelian monoid, written additively, and let $\mathrm{Fr}_{\mathbb{Z}}(M)$ be the free abelian group generated by the set M. Take R to be the subgroup of $\mathrm{Fr}_{\mathbb{Z}}(M)$ generated by all elements of the form

$$[x+y]-[x]-[y]$$

with $x, y \in M$; and define the *group completion* of M to be the quotient group

$$M_{\mathrm{gp}} = \mathrm{Fr}_{\mathbb{Z}}(M)/R\,,$$

with canonical projection

$$\begin{aligned} \pi : \mathrm{Fr}_{\mathbb{Z}}(M) &\longrightarrow M_{\mathrm{gp}} \\ [x] &\longmapsto \langle x \rangle\ . \end{aligned}$$

(i) Show that every homomorphism $f : M' \to M$ of abelian monoids induces a homomorphism $f_{\mathrm{gp}} : M'_{\mathrm{gp}} \to M_{\mathrm{gp}}$ of abelian groups.
(ii) Define a monoid homomorphism $\gamma : M \to M_{\mathrm{gp}}$ with the property that for every abelian group A it induces a bijection between the set of monoid homomorphisms from M to A and the set of group homomorphisms from M_{gp} to A.
(iii) Suppose that M is *cancellative*, that is, in M

$$x+z=y+z \quad \Longrightarrow \quad x=y\,.$$

Show that the relation

$$(x,y)\sim(x',y') \quad \Longleftrightarrow \quad x+y'=x'+y$$

on the monoid $M \times M$ is an equivalence relation, and that the set of equivalence classes $((x,y))$ forms an abelian group A such that there is an injective homomorphism from M to A. Deduce that γ is injective.
(iv) Give an example of an abelian monoid M such that γ is injective.

Question 2. **(a)** Let R be a ring. Show that for a nonzero right R-module M, the following are equivalent.

(i) M is irreducible.

(ii) M is cyclic and is generated by any one of its nonzero elements.

(iii) M is isomorphic to $R/\mathfrak{m}$ for some maximal right ideal $\mathfrak{m}$ of R.

(b) Let $\mathfrak{a}$ be a proper ideal of a ring R. Show that there is a simple ring S and a surjective ring homomorphism $\pi : R \twoheadrightarrow S$ with $\pi(\mathfrak{a}) = 0$.

Question 3. **(a)** (i) Show that every ring contains a maximal commutative subring.

(ii) Give an example (without proof) of a ring whose centre is not a maximal commutative subring.

(iii) Give an example (without proof) of a ring with more than one maximal commutative subring.

(b)(i) In the diagram

$$\begin{array}{ccc} & & M_2 \\ & & \downarrow{\scriptstyle \varphi_2} \\ M_1 & \xrightarrow{\varphi_1} & M'' \end{array}$$

of right R-modules and R-homomorphisms, let M_0 be the pullback.

(I) Prove that φ_1 and its induced map $\bar{\varphi}_1 : M_0 \to M_2$ have isomorphic kernels.

(II) Prove that there exists $\sigma : M_2 \to M_0$ with $\bar{\varphi}_1\sigma = \mathrm{id}_{M_2}$ if and only if there exists $\nu : M_2 \to M_1$ with $\varphi_1\nu = \varphi_2$. For the "if" direction, your proof should use the universal property of the pullback.

(ii) For $i = 1, 2$, let

$$0 \to M_i' \to M_i \to M'' \to 0$$

be a short exact sequence of right R-modules. Prove that if M'' is projective, then

$$M_1' \oplus M_2 \cong M_2' \oplus M_1.$$

(iii) For which values of $m \in \mathbb{Z}$ is $\mathbb{Z}/m\mathbb{Z}$ a projective $\mathbb{Z}$-module?

(iv) Give an example (without proof) to show that (b)(ii) above can fail to hold when M'' is not projective.

Question 4. **(a)** Let $f : R \to S$ be a ring homomorphism and let M be a right S-module.

(i) Show that if M is Artinian as a right R-module via f, then M is Artinian as a right S-module.

(ii) Show that the converse also holds when f is surjective.

(b) (i) Let S be a right Artinian ring. Show that if $a \in S$, then there is an integer k and some $b \in S$ with $a^k = a^{k+1}b$.

(ii) Deduce that if S is also a domain (that is, has no zero divisors), then every element has a right inverse.

(iii) Deduce further that if S is a domain, then it is a division ring.

(c) Deduce that in a right Artinian ring every prime twosided ideal is maximal.

Question 5. **(a)** Let F, K be fields with $F \subseteq K$. Suppose that L, M are subfields of K that contain F. For any subfield E of K that contains F, we denote the transcendence degree of E over F by tr.d.(E/F). Prove that

(i) tr.d.$(LM/F) \geq$ tr.d.(L/F).

(ii) tr.d.$(LM/F) \leq$ tr.d.$(L/F) +$ tr.d.(M/F).

(b) Suppose that L and M are quotient fields of integral domains R and S respectively. Suppose that K is a field that contains both L and M, and that F is a subfield in both L and M. Prove that L and M are linearly disjoint over F if and only if for every F-basis X of R and F-basis Y of S, the set $\{xy \mid x \in X,\ y \in Y\}$ is linearly independent over F.

END OF PAPER