PhD Qualifying Examination
Linear Algebra
Sem 2, 2006/2007
Time allowed: 2 hours

Answer **all** questions.

1. Show that if U is a subspace of $\mathbb{R}^n$, then there is a matrix A such that

$$U = \{\mathbf{u} \in \mathbb{R}^n : \ A\mathbf{u} = \mathbf{0}\}.$$

2. Let V be a finite dimensional vector space and let S and T be linear operators on V. Suppose that there exist bases $\mathcal{B}$ and $\mathcal{B}'$ of V such that $[S]_{\mathcal{B}} = [T]_{\mathcal{B}'}$. Prove that there is an invertible linear operator L on V such that $T = LSL^{-1}$.

3. Let A be a 4×4 complex matrix. Suppose that $\lambda_1 = 1$ and $\lambda_2 = 2$ are all the eigenvalues of A and

$$\dim V_{\lambda_1} + \dim V_{\lambda_2} = 3,$$

where for $i = 1, 2$, V_{λ_i} is the eigenspace of A corresponding to the eigenvalue λ_i.

(i) List all the possible Jordan canonical forms of A. Be sure that no two matrices in your list are similar to each other.

(ii) For each of the Jordan canonical forms in your list, state its minimal polynomial and characteristic polynomial.

4. Let V be a finite dimensional vector space and $T : V \to V$ a linear operator. Suppose that the subspace W of V is invariant under T, that is, $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$.

(i) Suppose that $\dim W = r$ and $\dim V = n$. Prove that V has a basis $\mathcal{B}$ such that

$$[T]_{\mathcal{B}} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is a $r \times r$ matrix, C is a $r \times (n-r)$ matrix and D is a $(n-r) \times (n-r)$ matrix.

(a) Define the restriction operator $T_W : W \to W$ by

$$T_W(\mathbf{w}) = T(\mathbf{w}), \quad \mathbf{w} \in W.$$

Prove that the minimal polynomial for T_W divides the minimal polynomial for T.

5. Let V be a finite dimensional real inner product space and T a self-adjoint operator on V such that

$$\langle T(\mathbf{v}), \mathbf{v} \rangle \geq 0.$$

(The operator T is called a *non-negative* operator.)

(i) Prove that all the eigenvalues of T are nonnegative.

(ii) Prove that there is a linear operator S on V such that $S^2 = T$.

6. If A is an $n \times n$ complex matrix, then its conjugate transpose is given by

$$A^* = \left(\overline{A}\right)^t.$$

If $A = A^*$, then we said that A is *Hermitian.* Let $\mathcal{H}$ be the space of all $n \times n$ Hermitian matrices and let B be a fixed $n \times n$ complex matrix. Define $T : \mathcal{H} \to \mathcal{H}$ by

$$T(A) = BAB^* \quad (A \in \mathcal{H}).$$

Prove that

$$\det T = |\det B|^{2n}.$$

End of paper