Ph.D. Qualifying Examination

Complex Analysis

Jan 2007

Convention: *Throughout this paper,* $\mathbb{C}$ *denotes the set of complex numbers. For* $a \in \mathbb{C}$ *and* $r > 0$*,* $D(a, r)$ *denotes the open disk* $\{z \in \mathbb{C} : |z - a| < r\}$*.*

1. (a) Find the number of roots of the equation $e^{-z} + z^2 - 9 = 0$ in the right half plane $\mathrm{Re} z > 0$. Justify your answer carefully.

   (b) Evaluate the improper integral

$$\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}.$$

2. (a) Find the Laurent series which represents the function

$$f(z) = \frac{1}{1 - z}$$

   in the region $\{z \ : \ |z| > 1\}$.

   (b) Suppose that f is analytic inside and on the simple closed curve C and that $|f(z) - 1| < 1$ for all z on C. Prove that f has no zeroes inside C.

3. Let $f : D(0; 1) \longrightarrow D(0; 1)$ be analytic. Prove that if there exist two distinct points α, β in the unit disk which are fixed points (i.e. $f(\alpha) = \alpha$, $f(\beta) = \beta$), then $f(z) = z$ for all z in $D(0; 1)$. (You may use without proof the fact that $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$, where $|a| < 1$, is an analytic automorphism of $D(0, 1)$).

4. Find an analytic isomorphism from the open half disk

$$U = \{z \in \mathbb{C} \ : \ |z - 3| < 2, \quad \text{and} \quad \mathrm{Im} z > 0\}$$

   to the unit disk $D(0, 1)$. (You may leave your result as a composition of functions).

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