Ph.D. Qualifying Examination

Analysis

Sem 2, 2006/2007

1. (a) Let E be a nonempty subset of $\mathbb{R}$ and suppose that $f_k,\ g_k : E \to \mathbb{R},\ k \in \mathbb{N}$, if

$$\left|\sum_{k=1}^{n} f_k(x)\right| \leq M < \infty$$

for $n \in \mathbb{N}$ and $x \in E$, and if $g_k \downarrow 0$ uniformly on E as $k \to \infty$, prove that $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly on E.

(Abel's formula: $\sum_{k=m}^{n} a_k b_k = A_{n,m} b_n - \sum_{k=m}^{n-1} A_{k,m}(b_{k+1} - b_k)$, where $A_{n,m} = \sum_{k=m}^{n} a_k$).

(b) Prove that, for each $x \in (0, 2\pi)$,

$$\left|\sum_{k=1}^{n} \cos(kx)\right| \leq \frac{1}{|\sin\frac{x}{2}|}$$

(Formula: $2 \sin \alpha \cos \beta = \sin(\alpha + \beta) + \sin(\alpha - \beta)$).

(c) Prove that if $a_k \downarrow 0$ as $k \to \infty$, then $\sum_{k=d}^{\infty} a_k \cos(kx)$ converges uniformly on any closed subinterval $[a, b]$ of $(0, 2\pi)$.

2. A metric space is called separable if it contains a countable dense subset. A subset K of a metric space is said to be compact if every open cover of K contains a finite subcover.

Prove that every compact metric space is separable.

3. $f : [a, b] \to \mathbb{R}$ be bounded.

(i) The oscillation of f on an interval J that intersects $[a, b]$ is defined to be

$$\Omega_f(J) := \sup_{x,y \in J \cap [a,b]} (f(x) - f(y)).$$

(ii) The oscillation of f at a point $t \in [a, b]$ is defined to be

$$\omega_f(t) := \lim_{h \to 0_+} \Omega_f((t - h, t + h)).$$

Prove that

(a) f is continuous at $t \in [a,b]$ if and only if $\omega_f(t) = 0$.

(b) let E represent the set of points of discontinuity of f in $[a,b]$. Prove that

$$E = \bigcup_{j=1}^{\infty} \left\{ t \in [a,b] : \omega_f(t) \geq \frac{1}{j} \right\}.$$

(c) For each $\varepsilon > 0$, let

$$H = \{t \in [a,b] : \omega_f(t) \geq \varepsilon\}.$$

Prove that H is compact.

(Hint: H is compact if and only if H is bounded and closed.)

(d) Let I be a closed subinterval of $[a,b]$ and $\varepsilon > 0$. If $\omega_f(t) < \varepsilon$ for all $t \in I$, prove that there exists $\delta > 0$ such that $\Omega_f(J) < \varepsilon$ for all closed subintervals J of I that satisfy $|J| < \delta$.

4. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-intergrable on $[a,b]$, and put

$$F_n(x) = \int_a^x f_n(t)dt, \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ which converges uniformly on $[a,b]$.

END OF PAPER