Ph.D. Qualifying Examination : Linear Algebra

Sem 1, 2006/2007

(1) Let V be a vector space of $n \times n$ matrices over $\mathbb{C}$. Let A be an element of V and let $f_A : V \to V$ be the map $f_A(X) = AX$. Prove that f_A is linear and that $\det(f_A) = (\det A)^n$.

(ii) Let V be a finite dimensional vector space over $\mathbb{C}$ with a non-degenerate scalar product $\langle\ ,\ \rangle$ and let f be a linear map of V into $\mathbb{C}$. Prove that there exists a unique element $v \in V$ such that $f(w) = \langle v, w \rangle$.

(iii) Let V be a finite dimensional vector space over $\mathbb{C}$ and let A be a linear operator. Let $f(x)$ be a polynomial such that $f(A) = 0$ and let $f(x) = (x - e_1)^{m_1}(x - e_2)^{m_2} \cdots (x - e_t)^{m_t}$, where $e_i \neq e_j$ for all $i \neq j$. Let V_i be the kernel of $(A - e_i I)^{m_i}$. Prove that V is a direct sum of the subspaces V_1, $V_2, \cdots, V_t$.

(iv) Let V be a finite dimensional vector space over $\mathbb{C}$ and let S be a set of linear operators of V. Suppose that the only subspaces invariant under S are 0 and V (a subspace W of V is said to be invariant under S if $BW \subseteq W$ for all B is S). Let X be a linear map such that $XA = AX$ for all A in S. Prove that either X is invertible, or X is the zero map.

(v) Two matrices A and B with integer entries are said to be similar to each other over $\mathbb{Z}$ if there exists a matrix P such that

   (a) both P and P^{-1} are integer matrices,
   (b) $A = PBP^{-1}$.

   Find all integer matrices (up to similarity over $\mathbb{Z}$) with characteristic polynomial $x^2 + 1$. Justify your answer.

End of Paper