Ph.D. Qualifying Examination

Analysis

Sem 1, 2006/2007

Answer All Questions

1. A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover.

   (a) Prove that if K is a compact subset in a metric space X, the K is closed and bounded. Is the converse true? Justify your answer.

   (b) Prove that if $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty.

2. Let $\delta : [0,1] \times [0,1] \to (0, \infty)$. Use Q.1(b) to show that there exist finite collections of nonoverlapping rectangles $\{I_k\}_{k=1}^n$ and points $\{x_k\}_{k=1}^n$ such that $x_k \in I_k \subseteq B(x_k, \delta(x_k))$ and $\overset{n}{\underset{k=1}{\cup}} I_k = [0,1] \times [0,1]$, where $B(x, \delta(x)) = \{y \in [0,1] \times [0,1]; d(x,y) < \delta(x)\}$ and d is the euclidean metric on $\mathbb{R}^2$.

3. (a) Prove that a function $f : (a,b) \to \mathbb{R}$ is uniformly continuous on (a,b) if and only if it can be extended to a function $\hat{f}$ that is continuous on $[a,b]$.

   (b) Let f and g be real-valued functions that are uniformly continuous on a compact set $D \subseteq \mathbb{R}$. Suppose that $g(x) \neq 0$ for all $x \in D$. Is $\dfrac{f}{g}$ uniformly continuous on D? Justify your answer.

4. The family $\mathcal{F}$ of functions from the metric space (S, d) to the metric space (T, ρ) is called *equicontinuous* on S if given any $\varepsilon > 0$ there is a $\delta > 0$ such that for every $f \in \mathcal{F}$, $\rho(f(x_1), f(x_2)) < \varepsilon$ whenever $d(x_1, x_2) < \delta$. Prove that if (S, d) is a compact metric space and the sequence $f_n : S \to T$ is equicontinuous on S and $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in S$ then the sequence $\{f_n\}$ converges uniformly to f on S.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2006-2007

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

Answer **ALL** questions.