Original

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 EXAMINATION 2005-2006

MA3501 Mathematical Methods in Engineering

April/May 2006 — Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **SEVEN (7)** printed pages, including three appendices.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

Question 1 [16 marks]

Solve, by the method of characteristics, the following 1^{st} order PDE:

$$u_t(x,t) + 3tu_x(x,t) = u(x,t), \quad -\infty < x < \infty, t > 0,$$

$$u(x,0) = \cos x, \qquad -\infty < x < \infty.$$

Question 2 [18 marks]

(a) Almost all high school students who intend to go to college take the SAT test. Debbie Sears is planning to take this test soon. Suppose the SAT score of all students who take this test with Debbie will have a normal distribution with a mean of 1016 and a standard deviation of 153. What should her score be on this test so that only 10% of all examinees score higher than she does?

(b) Because couples are deciding to have fewer children, the family size has decreased over the past few decades. According to the Bureau of the Census, the mean family size was 3.18 in 2000. A researcher wanted to check if the current mean family size is less than 3.18. A sample of 900 families taken this year by this researcher produced a mean family size of 3.16 with a standard deviation of 0.70. Using the 0.025 significance level, can we conclude that the mean family size has decreased since 2000?

Question 3 [16 marks]

(a) Prove that e^z is differentiable everywhere and $\frac{d}{dz}e^z = e^z$.

(b) **(i)** What is the value (in terms of t_1, t_2, x and y) of $\displaystyle\int_{t_1}^{t_2} \frac{y}{(x-t)^2 + y^2}\, dt$? (Do not need to justify your answer.)

(ii) Use Poisson's integral formula for the half-plane to find the solution of the following Laplace equation:

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad -\infty < x < \infty, \quad y > 0,$$

$$u(x,0) = a \quad \text{if } x < 0,$$

$$u(x,0) = b \quad \text{if } x > 0.$$

(iii) Find the value of $u(y,y)$, where $y > 0$.

Question 4 [17 marks]

(a) Let $f(z) = \frac{1}{(z+1)^2(z^2+1)}$.

(i) Find all the singular points of f.

(ii) Find $\displaystyle\int_{\gamma_1} f(z)\,dz$ and $\displaystyle\int_{\gamma_2} f(z)\,dz$, where
γ_1 is the circle with center $-\frac{1}{2}+\frac{i}{2}$ and radius 1;
γ_2 is the circle with center 0 and radius $\frac{1}{2}$.

(b) Let $g(z)$ be analytic inside a circle with centre 0 and radius 2. Prove that $g(0) = \frac{1}{2\pi}\displaystyle\int_0^{2\pi} g(e^{i\theta})\,d\theta$. (Hint: Use Cauchy's formula for integral)

Question 5 [17 marks]

(a) Solve the following heat equation by the method of separation of variables:

$$u_t(x,t) = u_{xx}(x,t), \qquad -1 < x < 1, t > 0,$$

$$u(-1,t) = u(1,t), u_x(-1,t) = u_x(1,t), \qquad t > 0,$$

$$u(x,0) = x+1, \qquad -1 < x < 1.$$

(b) Suppose v and w are solutions of the following Laplace equation:

$$u_{xx}(x,y) + u_{xx}(x,y) = 0, \qquad 0 < x < 1, 0 < y < 1,$$

$$u(x,0) = x, u(x,1) = 1-x, \qquad 0 < x < 1,$$

$$u(0,y) = y, u(1,y) = 1-y, \qquad 0 < y < 1.$$

Prove that $v(x,y) = w(x,y)$ for all $(x,y) \in [0,1]\times[0,1]$, i.e., the above Laplace equation with boundary conditions has a unique solution.

Question 6 [16 marks]

(a) Use the method of the Fourier transform to find the solution of the following PDE:

$$u_t(x,t) = tu_{xx}(x,t) \quad , \quad -\infty < x < \infty, t > 0,$$

$$u(x,0) = f(x).$$

(b) Let v be the solution of the following Laplace equation for the unit disk with center 0:

$$\frac{\partial^2 u}{dr^2} + \frac{1}{r}\frac{\partial u}{dr} + \frac{1}{r^2}\frac{\partial^2 u}{d\theta^2} = 0, \quad 0 < r < 1, 0 < \theta < 2\pi,$$

$$u(1,\theta) = \sin\theta, \qquad 0 < \theta < 2\pi.$$

Find the solution (in term of v) of the following Laplace equation for the semi-disk. Justify your answer.

$$\frac{\partial^2 u}{dr^2} + \frac{1}{r}\frac{\partial u}{dr} + \frac{1}{r^2}\frac{\partial^2 u}{d\theta^2} = 0, \quad 0 < r < 1, 0 < \theta < \pi,$$

$$u(1,\theta) = \sin\theta, \qquad 0 < \theta < \pi,$$

$$u(r,0) = 0, \qquad 0 \leq r < 1,$$

$$u(r,\pi) = 0, \qquad 0 \leq r < 1.$$

Three appendices attached: Formulae, Standard Normal Distribution Table and Boundary Value Problems.

END OF PAPER

Formulae

(A) $\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)x'(t)$

(B) Poisson's integral formula for the half-plane:
$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2}u(t,0)dt$$

(C) $\int_{-1}^{1}(x+1)\cos(n\pi x)dx = 0$

(D) $\int_{-1}^{1}(x+1)\sin(n\pi x)dx = (-1)^{n+1}\frac{2}{n\pi}$

(E) Fourier Transforms

Let $\mathcal{F}(f)(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}dx, \quad -\infty < w < \infty.$

$\mathcal{F}(f)(w)$ denoted by $\hat{f}(w)$

$\mathcal{F}(u(x,t))$ denoted by $\hat{u}(w,t)$.

$\mathcal{F}(u_t,(x,t)) = \frac{d}{dt}\ \hat{u}(w,t)$

$\mathcal{F}(u_{xx}(x,t)) = -w^2\ \hat{u}(w,t)$

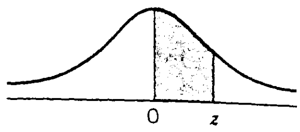
$$\mathcal{F}\left(\frac{e^{\frac{-x^2}{2t^2}}}{t}\right)(w) = e^{-\frac{t^2}{2}w}$$

$$\mathcal{F}^{-1}\left(\hat{f}(w)\hat{g}(w)\right) = (f*g)(x)$$

$$(f*g)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau$$

STANDARD NORMAL DISTRIBUTION TABLE

The entries in this table give the areas under the standard normal curve from 0 to *z*.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

BOUNDARY VALUE PROBLEMS for $X''(x) + \lambda X(x) = 0$

Boundary conditions	$X(0) = 0$ $X(L) = 0$	$X'(0) = 0$ $X'(L) = 0$	$X(-L) = X(L)$ $X'(-L) = X'(L)$
Eigenvalues λ_n	$(\frac{n\pi}{L})^2$ $n = 1, 2, 3, \ldots$	$(\frac{n\pi}{L})^2$ $n = 0, 1, 2, 3, \ldots$	$(\frac{n\pi}{L})^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin \frac{n\pi x}{L}$	$\cos \frac{n\pi x}{L}$	$\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$
Coefficients	$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$	$A_0 = \frac{1}{L} \int_0^L f(x) dx$ $A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$	$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx$ $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$ $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$

Boundary conditions	$X(0) = 0$ $X'(L) = 0$	$X'(0) = 0$ $X(L) = 0$
Eigenvalues λ_n	$\left[\frac{(2n-1)\pi}{2L}\right]^2$ $n = 1, 2, 3, \ldots$	$\left[\frac{(2n-1)\pi}{2L}\right]^2$ $n = 1, 2, 3, \ldots$
Eigenfunctions	$\sin \frac{(2n-1)\pi x}{2L}$	$\cos \frac{(2n-1)\pi x}{2L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{(2n-1)\pi x}{2L}$	$f(x) = \sum_{n=1}^{\infty} B_n \cos \frac{(2n-1)\pi x}{2L}$
Coefficients	$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx$	$B_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} dx$