Soloution of Exam 2006

Q 1 Example given in Lecture Note

Q 2 (a) shall find x such that

$$P(x \le X) = 10\% = 0.1$$

$$P\left(\frac{x-1016}{153} \le \frac{X-1016}{153}\right)$$

$$= P\left(\frac{X-1016}{153} \le Z\right) = 0.1$$

$$\therefore \quad \frac{X-1016}{153} = 1.28$$

$$x = 1.28(153) + 1016 = 1212$$

Her score should be 1212.

Q 2 (b)

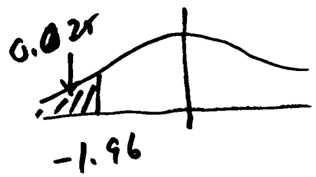
$H_0: \ \mu = 3.18$

$H_1: \ \mu < 3.18$

$n = 900, \ \bar{x} = 3.16, \ s = 0.70$
use s to estimate σ

[illegible]

$$z = \frac{\bar{x} - \mu_{\bar{x}}}{\sigma_{\bar{x}}} = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{3.16 - 3.18}{\frac{0.70}{\sqrt{900}}} = -0.86$$



-0.86 falls in accept region.

We can't conclude that the mean family size has decreased since 2000.

Q3 (a)

$e^z = e^x(\cos y + i \sin y)$

$u(x,y) = e^x \cos y$

$v(x,y) = e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$

$v_x = e^x \sin y$, $v_y = e^x \cos y$

$\therefore\ u_x = v_y$, $u_y = -v_x$

u_x, u_y, v_x, v_y are all continuous

$\therefore$ e^z is diff everywhere

$$\frac{d}{dz} e^z = u_x + i v_x = e^x \cos y + i e^x \sin y$$
$$= e^z$$

(b) (i) $$\int_{t_1}^{t_2} \frac{y}{(x-t)^2+y^2}\,dt = \tan^{-1}\left(\frac{y}{x-t_2}\right) - \tan^{-1}\left(\frac{y}{x-t_1}\right)$$

(ii) $$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y\,U(t)}{(x-t)^2+y^2}\,dt$$
$$= \frac{a}{\pi}\int_{-\infty}^{0} \frac{y}{(x-t)^2+y^2}\,dt + \frac{b}{\pi}\int_{0}^{\infty}\frac{y}{(x-t)^2+y^2}\,dt$$
$$= \frac{a}{\pi}\left[\tan^{-1}\frac{y}{x-0} - 0\right] + \frac{b}{\pi}\left[\pi - \tan^{-1}\frac{y}{x-0}\right]$$
$$= \frac{a-b}{\pi}\tan^{-1}\left(\frac{y}{x}\right) + b$$

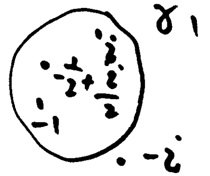
(iii)

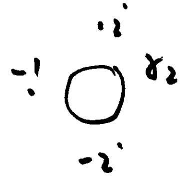
$$u(y,y) = \frac{a-b}{\pi}\tan^{-1}\frac{y}{y} + b$$
$$= \frac{a-b}{\pi}\,\frac{\pi}{4} + b$$
$$= \frac{a-b}{4} + b = \frac{1}{4}a + \frac{3}{4}b$$

Q4

(a)

(i) singular pts : $z=-1$, $z = i$, $z = -i$





(ii) $\mathrm{Res}(f, i) = g(i)$, where $g(z) = \dfrac{1}{(z+1)^2(z+i)}$

$$= \frac{1}{(i+1)^2(2i)} = -\frac{1}{4}i$$

$$\mathrm{Res}(f,-1) = \frac{g'(-1)}{1!} = \frac{1}{2}$$

where $g(z) = \dfrac{1}{z^2+1}$

$g'(z) = (-1)(z^2+1)^{-2}(2z)$

$g'(-1) = \frac{1}{2}$

$$\int_{\gamma_1} f(z)\,dz = 2\pi i\,[\operatorname{Res}(f, i) + \operatorname{Res}(f, -1)]$$
$$= 2\pi i\left[-\tfrac{1}{4} i + \tfrac{1}{2}\right]$$
$$= \pi\left(\tfrac{1}{2} + i\right)$$

$$\int_{\gamma_2} f(z)\,dz = 0$$

(b) $$g(0) = \frac{1}{2\pi i}\int_\gamma \frac{g(z)}{z}\,dz$$

where γ is the unit circle with centre 0.

Let $z = e^{i\theta}$. Then $dz = i e^{i\theta} d\theta$

$$g(0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(e^{i\theta})}{e^{i\theta}}\, i e^{i\theta} d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})\,d\theta$$

Q5

(a) Let $u(x,t) = X(x)T(t)$

We get

$$X''(x) + \lambda X(x) = 0, \quad -1 < x < 1$$
$$X(-1) = X(1) \quad X'(-1) = X'(1)$$

$$T'(t) + \lambda T(t) = 0$$

From table of BVP, get

$\lambda_n = (n\pi)^2, \quad n = 0, 1, 2, \ldots$

$X_n(x) = \cos(n\pi x)$

$X_n^*(x) = \sin(n\pi x)$

For each $\lambda_n = (n\pi)^2, \quad n = 0, 1, 2, \ldots$

$T_n'(t) + \lambda_n T_n(t) = 0$

get $T_n(t) = e^{(n\pi)^2 t}$ for $n = 0, 1, 2, \ldots$

$$\therefore u(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\, e^{(n\pi)^2 t} + \sum_{n=0}^{\infty} B_n \sin(n\pi x)\, e^{(n\pi)^2 t}$$

$$= \sum_{n=0}^{\infty} A_n \cos(n\pi x)\, e^{(n\pi)^2 t} + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{(n\pi)^2 t}$$

$$u(x,0) = x+1 = \sum_{n=0}^{\infty} A_n \cos(n\pi x) + \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

$$A_0 = \frac{1}{2}\int_{-1}^{1} (x+1)\,dx = 1$$

$$A_n = \int_{-1}^{1} (x+1)\cos(n\pi x)\,dx = 0, \quad n = 1, 2 \ldots$$

$$B_n = \int_{-1}^{1} (x+1)\sin(n\pi x)\,dx$$

$$= (-1)^{n+1}\frac{2}{n\pi}, \quad n = 1, 2 \ldots$$

$$u(x,t) = 1 + \sum_{n=1}^{\infty} (-1)^{n+1}\frac{2}{n\pi}\sin(n\pi x)\, e^{-n^2\pi^2 t}$$

5(b) Let $p(x,y) = v(x,y) - w(x,y)$

Then $p(x,y) = 0$ on boundary

$\therefore$ $p(x,y) = 0$ in $[0,1]\times[0,1]$

$\therefore$ $v(x,y) = w(x,y)$ in $[0,1]\times[0,1]$

6(a)

$$F(u_t(x,t)) = t\, F(u_{xx}(x,t))$$

$$F(u(x,0)) = F(f(x))$$

$\boxed{\text{Let } F(u(x,t)) = \hat{u}(w,t)}$

get

$$\frac{d}{dt}\hat{u}(w,t) = -t w^2\, \hat{u}(w,t)$$

$$\hat{u}(w,0) = \hat{f}(w)$$

$\boxed{\text{Let } F(f(x)) = \hat{f}(w)}$

Solve $\frac{d}{dt}\hat{u}(w,t) = -t w^2\, \hat{u}(w,t)$

$$\frac{1}{\hat{u}(w,t)}\, d\hat{u}(w,t) = -t w^2\, dt$$

$$\therefore \ln \hat{u}(w,t) = -w^2\frac{t^2}{2} + C$$

$$\hat{u}(w,t) = e^{C}\, e^{-\frac{t^2}{2} w^2}$$

use $\hat{u}(w,0) = \hat{f}(w)$, get

$$e^{c} = \hat{f}(w)$$

$$\therefore \quad \hat{u}(w,t) = \hat{f}(w)\, e^{-\frac{t^2}{2}w^2}$$

$$= \hat{f}(w)\, F\left(\frac{e^{-\frac{x^2}{2t^2}}}{t}\right)$$

$$\therefore \quad u(x,t) = f * \left(\frac{e^{-\frac{x^2}{2t^2}}}{t}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\tau)\, \frac{e^{-\frac{\tau^2}{2t^2}}}{t}\, d\tau$$

$$= \frac{1}{t\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-\tau)\, e^{-\frac{\tau^2}{2t^2}}\, d\tau$$

this take in the table

$$F\left(\frac{e^{\frac{-x^2}{2t^2}}}{t}\right)(w) = e^{-\frac{t^2}{2}w^2}$$

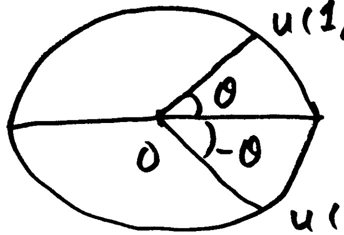
6(b)

Note that

$$u(r,0) = 0, \quad 0 \le r < 1$$

$$u(r,\pi) = 0, \quad 0 \le r < 1$$

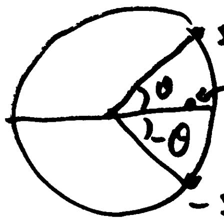
since



$u(1,\theta) = \sin\theta$

$u(1,\theta) = \sin(-\theta)$

$= -\sin\theta$

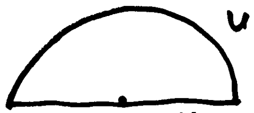


$\sin\theta$

temperature $= 0$

$-\sin\theta$

$\therefore$ v is also a solution of



$u(1,\theta) = \sin\theta$

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