NATIONAL UNIVERSITY OF SINGAPORE

FACULTY OF SCIENCE

SEMESTER 1 EXAMINATION 2005-2006

MA5203 Graduate Algebra I

November 2005 – Time allowed : 2.5 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FOUR (4)** printed pages.
2. Answer not more than **FOUR (4)** questions.
3. Maximum marks will be allocated as follows:

Your best 3 answers	:	27% each	81%
Next best answer	:	19%	19%

			100%

4. Results *proved* in lectures or tutorial assignments that you use should be stated clearly but need not be proved.
5. The symbol R always refers to a ring (with 1), assumed arbitrary, unless otherwise stated.

Question 1. **(a)** Show that any non-finitely-generated subgroup of a group G must be contained in a maximal non-finitely-generated subgroup of G.

(b) Show that if M is a right R-module M then either M is Noetherian or else contains a non-finitely-generated submodule M' such that M/M' is Noetherian.

(c) Let M be a nonzero finitely generated Artinian right R-module. Show that M has a maximal nonzero Noetherian quotient module M'' (in the sense that any factorization of the canonical projection $M \twoheadrightarrow M''$ as

$$M \twoheadrightarrow Q \twoheadrightarrow M'',$$

with Q Noetherian, has the R-map $Q \twoheadrightarrow M''$ an isomorphism).

Question 2. Let S be a commutative ring.

(a) For any right S-module M, let $\cdot s : M \to M$ be multiplication by $s \in S$, and define

$$\begin{aligned} \mathrm{Ann}(M) &= \{s \in S \mid \cdot s = 0\}, \\ \mathrm{Mon}_M &= \{s \in S \mid \cdot s \text{ is a monomorphism}\}. \end{aligned}$$

Briefly prove the following.

(i) $\mathrm{Ann}(M)$ is an ideal of S and M is a right module over $S/\mathrm{Ann}(M)$.

(ii) If N is a right S-module such that $\mathrm{Ann}(M) \cap \mathrm{Mon}_N \neq \emptyset$, then the group $\mathrm{Hom}(M_S, N_S)$ is zero.

(b) Let R be a commutative ring (you may assume without proof that R has invariant basis number).

(i) Write down (with proof) a free generating set for $\mathrm{End}(R^m_R)$ as a left R-module.

(ii) Write down a left R-module isomorphism from $M_m(R)$ to $\mathrm{End}(R^m_R)$.

(iii) Show that if $A \in M_m(R)$ then there is a polynomial $f(x) \in R[x]$ of degree at most m^2 that vanishes on A; that is, $f(A)$ is the zero matrix.

Now suppose further that $g(x)$ vanishes on $B \in M_n(R)$, where $f(x), g(x) \in R[x]$ are coprime.

(iv) Let M be the right $R[x]$-module R^m on which x acts as A, and let N be the right $R[x]$-module R^n on which x acts as B. Show that $\mathrm{Hom}(M_{R[x]}, N_{R[x]}) = 0$.

Question 3. Let R be a (commutative) domain.

(a)(i) Show that if I and J are nonzero ideals of R then $IJ \neq 0$.

(ii) Show that R_R is indecomposable.

(b) Let $a \in R$. Consider the diagram below in which each map of right R-modules is the inclusion map.

$$\begin{array}{ccc} aR & \longrightarrow & R \\ \downarrow & & \\ R & & \end{array}$$

(i) Describe the universal property for a pushout N of this diagram, and show that it exists.

(ii) By using the map $(x, y) \longmapsto x + y$, or otherwise, define a split epimorphism of R-modules from N to R having kernel isomorphic to R/aR.

(iii) Find necessary and sufficient conditions on a for N to be a projective R-module.

Question 4. Let R be a commutative ring, and let $\mathfrak{p}$ be a prime ideal in R.

(a) Briefly outline the argument that shows that the localization $R_\mathfrak{p}$ is a flat R-module.

(b) Let $\mathfrak{a}$ be an ideal of R. Show that the following are equivalent.

(i) $\mathfrak{a}$ is not contained in $\mathfrak{p}$.

(ii) $(R/\mathfrak{a}) \otimes_R R_\mathfrak{p} = 0$.

(iii) $\mathfrak{a}_\mathfrak{p} = R_\mathfrak{p}$.

(c) Use (b) to show that every proper ideal of R is contained in a prime ideal. Briefly indicate a more common way of obtaining this conclusion.

Question 5. (a) Let $\alpha : L \to M$ and $\beta : M \to N$ be morphisms in an abelian category $\mathcal{A}$. By means of the Five Lemma, or otherwise, show that the following are equivalent.

(i)

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

is a short exact sequence.

(ii) α is a monomorphism and $(N, \beta) = \text{Coker}\alpha$.

(iii) β is an epimorphism and $(L, \alpha) = \text{Ker}\beta$.

Give an example to show that the implication (ii) $\Rightarrow$ (i) can fail in an additive category.

(b) Let $\mathcal{C}$ be a full subcategory of an abelian category $\mathcal{A}$.

(i) Show that $\mathcal{C}$ is additive if and only if the zero object of $\mathcal{A}$ lies in $\mathcal{C}$ and $\mathcal{C}$ is closed under direct sum.

(ii) Suppose that $\mathcal{C}$ is additive and, for any morphism $\lambda : L \to M$ in $\mathcal{C}$, the objects $\text{Ker}\lambda$ and $\text{Coker}\lambda$ of $\mathcal{A}$ are both in $\mathcal{C}$. Show that $\mathcal{C}$ is abelian.

(c) By considering torsion, or otherwise, give an example of a full subcategory of the category $\textsc{Ab}$ of abelian groups that is:

(i) additive but not abelian; and

(ii) abelian.

END OF PAPER