Graduate Qualifying Examination : Linear Algebra
Sem 2, 2005/2006

(1) Let A be a 4×4 matrix with complex entries. Suppose that $A^m = 0$ for some $m \in \mathbb{N}$. Prove that
 (i) $A^4 = 0$,
 (ii) there exists a 4×4 matrix B such that $B^2 = I + A$.

(2) Let V be a vector space of finite dimensional over $\mathbb{R}$ and let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two basis of V. Prove that $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same cardinality. Does our result hold for $P_\infty(\mathbb{R})$ (the infinite dimensional vector space of all polynomials over $\mathbb{R}$) ? Justify your answer.

(3) Let W be a subspace of an inner product space V. A best approximation to $v \in V$ by vectors in W is a vector $w \in W$ such that

$$||v - w|| \leq ||v - u|| \text{ for every vector } u \text{ in } W.$$

Prove that $w \in W$ is a best approximation of $v \in V$ by vectors in W if and only if $v - w$ is orthogonal to every vector in W. Let w_1 and w_2 be best approximation to $v \in V$ by vectors in W. Determine whether $w_1 = w_2$. Justify your answer.

(4) Let $M_4(\mathbb{R})$ be the set of all 4×4 matrices with real entries. Find a function $d : M_4(\mathbb{R}) \to \mathbb{R}$ such that
 (i) for each $A \in M_4(\mathbb{R})$, $d(E_{ij}A) = -d(A)$, $d(E_{ij}(c)A) = d(A)$, $d(E_i(c)A) = cd(A)$, where E_{ij} $(i \neq j)$ is the matrix obtained from I_4 by interchanging row i and j of I_4, $E_{ij}(c)$ $(i \neq j)$ is the matrix obtained from I_4 by adding c times of row j to row i, and $E_i(c)$ arises from I_4 on multiplying row i of I_4 by c,
 (ii) $d(I_4) = 1$, where I_4 is the 4×4 identity matrix.

Is $d : M_4(\mathbb{R}) \to \mathbb{R}$ unique ? Justify your answer.

(5) Let A be an $n \times n$ matrix. Determine whether A and A^t are similar to each other over $\mathbb{C}$, where A^t is the transpose of A. Justify your answer.

The End