Ph.D. Qualifying Examination
Complex Analysis
Semester 2, 2005/2006

1. (a) Suppose that f is an entire function such that

$$|f(z)| \leq |\, e^z| \quad \text{for all } z \in \mathbb{C},$$

and $f(0) = 1/2$. Determine f. Justify your answer carefully.

(b) Suppose that $g(z)$ is an entire function and $g(z) = g(3z)$ for all $z \in \mathbb{C}$. Prove that $g(z)$ is constant. Justify your answer carefully.

2. (a) Identify the singular points of the function

$$f(z) = \frac{e^z - 1}{z^3 \sin z}$$

and classify each as a removable singularity, an essential singularity or a pole (specify the order of the pole).

(b) Use the Cauchy integral formula to evaluate

$$\int_0^{2\pi} \frac{1}{13 + 5\sin\theta}\, d\theta.$$

3. Suppose that f is analytic in $|z| < 1$, $|f(z)| \leq M$ for $|z| < 1$, and $f(0) = 0$. Show that $|f(z)| \leq M|z|$ for $|z| < 1$. (State clearly any results used.)

4. Find an analytic isomorphism from the complement of the arc

$$\{z \in \mathbb{C} \cup \{\infty\} \ : \ |z| = 1, \mathrm{Re} z \geq 0\}$$

in $\mathbb{C} \cup \{\infty\}$ to the open unit disk centred at the origin.

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