

Ph.D. Qualifying Examination
Analysis
Sem 2, 2005/2006

Do All Questions

1. (i) Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X . Show that the sequence $\{d(p_n, q_n)\}$ converges.
(ii) Let $\{E_n\}$ be a sequence of closed and bounded sets in a complete metric space. If $E_n \supset E_{n+1}$ for all n and $\lim_{n \rightarrow \infty} \text{diam } E_n = 0$, prove that $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point. Can the condition "complete" be omitted? Justify your answer.
2. (i) Let d be a discrete metric defined on \mathbb{R} . What sets are open in (\mathbb{R}, d) ? What functions are uniformly continuous on (\mathbb{R}, d) ? Justify your answers.
(ii) A metric space is called separable if it contains a countable dense subset. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.
3. (i) Let $B[0, 1]$ be the space of all bounded functions defined on $[0, 1]$. Give a norm $\|\cdot\|$ defined on $B[0, 1]$ such that $\|f_n - f\| \rightarrow 0$ as $n \rightarrow \infty$ if and only if $\{f_n\}$ converges to f uniformly on $[0, 1]$.
(ii) Suppose $\{f_n\}$ converges to f uniformly on $[0, 1]$ and $\lim_{t \rightarrow x} f_n(t)$ exists for each n . Prove that
 - (i) $\lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$ exists, and
 - (ii) $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$.
4. (a) Let f be an increasing function defined on (a, b) . Prove that
 - (i) $f(x+)$ and $f(x-)$ exist at every point of x in (a, b) ,
 - (ii) the set of points in (a, b) at which f is discontinuous is at most countable.(b) If f is continuous on $[0, 1]$ and if
$$\int_0^1 f(x)x^n dx = 0, \quad n = 0, 1, 2, \dots,$$
prove that $f(x) = 0$ on $[0, 1]$.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2005-2006

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 4 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.