PhD Qualifying Exam
Algebra
Sem 2, 2005/2006

Answer all questions.
Each question carries 25 marks.

(1) (a) Prove that a group of order 12 either has a normal subgroup of order 3, or is isomorphic to A_4, the alternating group on 4 letters. [10 marks]

(b) Show that any simple group acting on a set of n elements is isomorphic to a subgroup of A_n, the alternating group on n letters. [15 marks]

(2) Let R be a ring, not necessarily commutative and not necessarily containing the multiplicative identity. Prove that if $R[X]$ is a principal ideal domain, then R is a field. [25 marks]

(3) Let K be the splitting field of $X^4 - 2$ over $\mathbb{Q}$. Find all intermediate fields between $\mathbb{Q}$ and K. [25 marks]

(4) Let R be a ring with multiplicative identity, and let M be a left R-module. Show that the following statements are equivalent: [25 marks]

(a) Every submodule of M is finitely generated.
(b) Whenever $N_1 \subseteq N_2 \subseteq \cdots$ is an ascending chain of submodules of M, there is an integer k such that $N_l = N_k$ for all $l \geq k$.
(c) Every non-empty collection of submodules of M has a maximal element (with respect to inclusion).

The End