Ph.D Qualifying Examination : Linear Algebra
2005/2006, Sem 1

(i) Let A and B be two $n \times n$ matrices. Suppose that $AB = BA$. Prove that there exists a basis $\mathcal{B}$ such that both $[A]_{\mathcal{B}}$ and $[B]_{\mathcal{B}}$ are upper triangular.

(ii) Let A and B be two $n \times n$ matrices. A and B are *similar* to each other over F if there exists an invertible $n \times n$ matrix P with entries in F such that $PA = BP$.

(a) Find all 7×7 matrices (up to similarity over $\mathbb{C}$) with minimal polynomial $(x^2 + 2x + 1)(x - 2)$. Justify your answer.

(b) Find all 2×2 matrices (up to similarity over $\mathbb{Z}$) with characteristic polynomial $(x^2 + 2x + 1)$. Justify your answer.

(iii) Let A and B be two $n \times n$ matrices. Prove that $\det AB = \det A \det B$.

(iv) Let V be an n-dimensional vector space over a field F and let $f : V \times V \to F$ be a bilinear form. Suppose that f is *nondegenerate* (if $f(x, v) = 0$ for all $x \in V$, then $v = 0$) and $f(u, u) = 0$ for all $u \in V$. Prove that $n = 2m$ is even and that there exists a basis $\{e_1, e_2, \cdots, e_{2m}\}$ such that

(a) $f(e_i, e_{m+i}) = 1$ for $1 \leq i \leq m$,

(b) $f(e_i, e_j) = 0$ if $|i - j| \neq m$.