Ph.D. Qualifying Examination
Analysis
Sem 1, 2005/2006

1. (a) Give an example of an open cover of $(0,1)$ which has no finite subcover.

   (b) A subset K of a metric space X is said to be compact if every open cover of K contains a finite subcover. Prove that every closed subset (relative to X) of a compact set is compact.

   (c) Let E_n, $n = 1,2,3,...$ be a sequence of countable sets, and $S = \bigcup_{n=1}^{\infty} E_n$. Prove that S is countable.

2. (a) Suppose (X,d) is a complete metric space and $\emptyset \neq A_n \subseteq X$ is closed for $n = 1,2,3,...$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ with $\lim_{n\to\infty} d(A_n) = 0$. Prove that $\bigcap_{n=1}^{\infty} A_n$ is a singleton set.

   (b) Let (X,d) be a metric space. Prove that $f : X \to \mathbb{R}$ is continuous if and only if for each open set G in $\mathbb{R}$, $f^{-1}(G)$ is open in X.

3. (a) Prove that $\sum_{k=1}^{\infty} \frac{\sin kx}{k^2}$ is uniformly convergent on $(-\infty,\infty)$.

   (b) Consider the sequence $\{S_n(x)\}$ defined on $[0,1]$ by

   $$S_n(x) = \begin{cases} n - n^2 x & \text{if } x \in (0, \frac{1}{n}) \\ 0 & \text{otherwise.} \end{cases}$$

   Does $\{S_n(x)\}$ converge uniformly on $[0,1]$? Justify your answer.

   (c) Is the uniform limit of a sequence of differentiable functions on $[-1,1]$ differentiable on $[-1,1]$? Justify your answer.

4. Let $f : [a,b] \to \mathbb{R}$. Then f is said to be regulated if for each $x \in [a,b]$, $\lim_{t\to x-} f(t)$ and $\lim_{t\to x+} f(t)$ exist.

   Use the Heine-Borel open covering theorem to prove that if f is regulated on $[a,b]$, then for each $\epsilon > 0$, there exists a finite sequence $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that for each $i = 1,2,...,n$ and any two points t', t'' with $t_{i-1} < t' < t'' < t_i$, we have

   $$|f(t'') - f(t')| \leq \epsilon.$$

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2005-2006

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.