Algebra, 2005/2006, Sem 1

Answer all questions. Each question carries 25 marks.

(1) Classify all groups of order 8 up to isomorphism.

(2) Prove or disprove each of the following statements:

(a) A field is a Euclidean domain.

(b) If R is a Euclidean domain but not a field, and S is a subring of R with multiplicative identity, then S is the unique factorization domain.

(3) For each of the following polynomials $f(X)$, find the degree of K over $\mathbb{Q}$, where K is the splitting field of $f(X)$.

(a) $f(X) = X^4 - 1$;
(b) $f(X) = X^3 - 1$;
(c) $f(X) = X^4 - 2$;
(d) $f(X) = X^3 - 2$.

(4) Let R be a ring with 1. A *simple* left R-module M is a left R-module such that $|M| > 1$ and if N is a submodule of M, then either $N = M$ or $N = \{0\}$.

(a) Let I be a maximal left ideal of R. Show that R/I is a simple R-module.
(b) Let m be a nonzero element of a simple left R-module M. Prove that:
(i) $Rm := \{rm \mid r \in R\}$ equals M;
(ii) $\operatorname{Ann}(m) := \{r \in R \mid rm = 0\}$ is a maximal left ideal of R;
(iii) $R/\operatorname{Ann}(m) \cong M$ as left R-modules .