Ph.D. Qualifying Examination

Linear Algebra

Sem 2, 2004/2005

Time allowed: 90 mins

1. Let
$$V = \{a_0 + a_1 x + \cdots + a_n x^n : \ n \geq 0, \ a_0, ..., a_n \in \mathbb{R}\}$$
be the vector space of all real polynomials and let $L(V)$ be the space of all linear operators $T : V \to V$. For each $m \geq 1$, define $T_m : V \to V$ by
$$T_m[f(x)] = \frac{d^m}{dx^m} f(x).$$
Prove that for any positive integer n, the set $\{T_1, T_2, ..., T_n\}$ is linearly independent in $L(V)$.

2. Let V be a vector space and $T : V \to V$ a linear operator. Suppose that
$$V = R(T) \oplus W$$
where $R(T)$ is the range of T, and W is a T-invariant subspace of V, that is, $T(\mathbf{w}) \in W$ for each $\mathbf{w} \in W$.

   (a) Prove that $W \subseteq \mathrm{Ker}(T)$.

   (b) Prove that if V is finite dimensional, then $W = \mathrm{Ker}(T)$.

   (c) Show by example that the conclusion of (b) is not necessarily true if V is not finite dimensional.

3. Let T be a self-adjoint linear operator on a finite dimensional complex inner product space V, that is,
$$\langle T(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, T(\mathbf{v}) \rangle, \qquad \forall \mathbf{u}, \mathbf{v} \in V.$$

   (a) Prove that all the eigenvalues of T are real.

   (b) Prove that if U is a T-invariant subspace of V, then so is its orthogonal complement $U^{\perp} = \{\mathbf{v} \in V : \ \langle \mathbf{v}, \mathbf{u} \rangle = 0 \ \forall \mathbf{u} \in U\}$.

   (c) Prove that T is diagonalizable.

4. If A is a square complex matrix with characteristic polynomial

$$c(x) = (x-1)^2(x+2)^2x^4$$

and minimal polynomial

$$m(x) = (x-1)(x+2)^2x^3,$$

what is the Jordan form for A?

5. Let $V = \mathrm{M}_n(\mathbb{R})$ be the space of all $n \times n$ real matrices and let A be a fixed $n \times n$ matrix. Define $T : V \to V$ by

$$T(X) = AX \qquad (X \in V).$$

Is it true that the linear operator T and the matrix A have the same eigenvalues? Justify your answer.

End of paper