Ph.D. Qualifying Examination
Complex Analysis

1. (a) Let a function $f(z)$ be analytic in a domain D. Show that if there exist real constants α, β and γ, not all zero, such that

$$\alpha \operatorname{Re} f(z) + \beta \operatorname{Im} f(z) = \gamma$$

throughout D, then $f(z)$ must be constant in D.

(b) Find the values, if any, of the positive integer n such that the function

$$h(x, y) = x^n - y^n,$$

where $x, y \in \mathbb{R}$, is harmonic in $\mathbb{C}$.

2. (a) Let n be an integer. Evaluate the integral

$$\int_C \frac{\cosh(nz)}{z}\, dz,$$

where C is the circle $|z| = 1$ oriented in the counterclockwise direction. Hence or otherwise, find the value of

$$\int_0^{\pi} \cosh(n\cos\theta)\cos(n\sin\theta)\, d\theta.$$

(b) Determine all the singular points, if any, of the function

$$f(z) = \frac{\sin z + z - \pi}{z(z-\pi)^4},$$

and for each singular point, identify its type and find the residue of f at it.

3. Use Cauchy's residue theorem to evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2}\, dx.$$

Justify your steps.

4. Consider the function

$$F(z) = z^2 e^{-z} + 6z - 5.$$

Let C be the positively oriented contour formed by the right half of the circle $|z| = 2$ and the straight line segment from $z = 2i$ to $z = -2i$.

(i) Determine the number of zeros, counting multiplicities, of $F(z)$ inside C.

(ii) Does $F(z)$ have any real zeros inside C? Justify your answer.

5. Find a linear fractional transformation that maps the lower half plane $\operatorname{Im} z < 0$ onto the disk $|w + 1| < 2$.

END OF PAPER