Ph.D. Qualifying Examination

Analysis

Sem 2, 2004/2005

Notation.

$(\mathbb{R}^n, \|\cdot\|_p)$ denotes the n-dimensional Euclidean space with norm $\|x\|_p = (\sum_{k=1}^{n} |x_k|^p)^{\frac{1}{p}}$, where $p \geq 1$.

$(R^n, \|\cdot\|_\infty)$ denotes the n-dimensional Euclidean space with norm $\|x\|_\infty = \max\{|x_k|; k = 1, 2, ..., n\}$.

$(\ell_p, \| \ \|_p)$ denotes the norm space of real sequences $x = (x_k)_{k=1}^{\infty}$ with norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}} < \infty$, where $p \geq 1$.

$(\ell_\infty, \|\cdot\|_\infty)$ denotes the norm space of bounded real sequences $x = (x_k)_{k=1}^{\infty}$ with norm $\|x\|_\infty = \sup\{|x_k| : k = 1, 2, ...\}$.

1. (a) Let $K \subset \mathbb{R}$ consists of 0 and the numbers $\frac{1}{n}$, for $n = 1, 2, 3, \ldots$. Prove that every open cover of K contains a finite subcover.

   (b) A metric space is called separable if it contains a countable dense subset. Show that $\mathbb{R}^2$ is separable.

   (c) A collection $\{V_\alpha\}$ of open sets of a metric space X is said to be a base for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α.

   Prove that every separable metric space has a countable base.

2. If E is a nonempty subset of a metric space (X, d), define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that

$$\rho_E(x) \leq d(x, y) + d(y, z)$$

for all $z \in E$ and all x, $y \in X$.

(b) Prove that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y) \quad \text{for all } x, y \in X.$$

(c) Suppose K and F are disjoint sets in X, K is compact and F is closed. Prove that

(i) ρ_F is a continuous function on the compact set K; and

(ii) there exists $\delta > 0$ such that $d(p, q) > \delta$ for all $p \in K$ and all $q \in F$.

Show that the statement (ii) may fail for two disjoint closed sets if neither is compact.

3. (a) Discuss the convergence, both pointwise and uniform, of

$$S_n(x) = \frac{1 - x^n}{1 - x}, \; n = 1, 2, ...$$

on $(-1, 1)$.

(b) Suppose K is compact and

(α) $\{g_n\}$ is a sequence of continuous functions on K,

(β) $\{g_n\}$ converges to 0 on K,

(γ) $g_n(x) \geq g_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, ...$

Let $\varepsilon > 0$ be given and for $n = 1, 2, 3, ...$, K_n the set of all $x \in K$ with $g_n(x) \geq \varepsilon$.

Prove that

(i) each K_n is closed and compact;

(ii) $\bigcap_{n=1}^{\infty} K_n$ is empty; and

(iii) there exists N such that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and all $n \geq N$.

4. (a) Let $x \in \mathbb{R}^n$, $x^{(m)} \in \mathbb{R}^n$, $m = 1, 2, ...$, prove that

   (i) $\|x^{(m)}\|_p \to 0$ as $m \to \infty$ iff for each k, $x_k^{(m)} \to 0$ as $m \to \infty$, where $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ..., x_n^{(m)})$; and

   (ii) $\|x\|_\infty = \lim_{p \to \infty} \|x\|_p$.

   (b) (i) Prove that $\ell_p \subset \ell_q$ if $1 \leq p \leq q \leq \infty$.

   (ii) Is (a)(i) true for $x^{(m)} \in \ell_p$, $m = 1, 2, ...$, where $x^{(m)} = (x_1^{(m)}, x_2^{(m)}, ...)$? Justify your answer.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2004-2005

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.