Qualifying Examination
Linear Algebra
Time allowed: 90 mins

Answer **all** questions.

1. (a) Let W be a subspace of an inner product space V. Define the orthogonal complement of W.

   (b) Let $\mathbb{R}^n$ be given the Euclidean inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} u_i v_i$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{pmatrix}.$$

Let A be a $m \times n$ real matrix. Prove that the row space of A is the orthogonal complement of the solution space of

$$A \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

2. Let V be a finite dimensional vector space and W a subspace of V. Suppose that $\dim W = r$, $\dim V = n$ and $r < n$. Let $\mathcal{F}$ be the family of all $(n-1)$-dimensional subspaces of V which contain W. Prove that

$$W = \bigcap_{U \in \mathcal{F}} U.$$

3. Let V be a finite dimensional vector space and T a linear operator on V. Prove that if

$$\operatorname{rank}(T^2) = \operatorname{rank}(T),$$

then

$$V = \operatorname{Ker}(T) \oplus \operatorname{Range}(T).$$

4. (a) State the Cayley-Hamilton Theorem.

(b) Let A be the 4×4 real matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

(i) Calculate the rank of A.

(ii) Compute A^2 and A^3.

(iii) Find the minimal polynomial and characteristic polynomial of A.

(iv) Is A diagonalizable? Justify your answer.

5. Let A be a nonzero $n \times n$ complex matrix. If $1 \leq r \leq n$, an $r \times r$ *submatrix* of A is any $r \times r$ matrix obtained by deleting $(n - r)$ rows and $(n - r)$ columns of A. The *determinant rank* of A is the largest positive integer r such that some $r \times r$ submatrix of A has a nonzero determinant. Prove that the determinant rank of A is equal to the rank of A.

End of paper