Ph.D. Qualifying Examination

Complex Analysis

1. (a) Determine the isolated singular points of the function $f(z) = \dfrac{\pi e^z + iz}{(z^2+\pi^2)^2}$. Classify each singular point as a removable singular point, an essential singular point or a pole of specific order. Evaluate also the residue of $f(z)$ at each of the singular points.

   (b) Let $g(z)$ be an entire function such that

$$g(0) = 0, \qquad g'(0) = 0, \qquad g''(0) = 6, \qquad g'''(0) = 8.$$

   Evaluate the residue of the function $\dfrac{1}{g(z)}$ at $z = 0$. Justify your answer.

2. Let $\mathbf{D}$ denote the standard unit open ball in the complex plane $\mathbf{C}$, i.e., $\mathbf{D} = \{z \in \mathbf{C} \,|\, |z| < 1\}$, and let $\overline{\mathbf{D}}$ denote its closure in $\mathbf{C}$, i.e., $\overline{\mathbf{D}} = \{z \in \mathbf{C} \,|\, |z| \leq 1\}$. Suppose a function f is analytic on $\overline{\mathbf{D}}$ except for some poles in $\mathbf{D}$. Show that the number of poles of f in $\mathbf{D}$ must be finite.

3. Use the Cauchy residue theorem to evaluate the improper integral

$$\int_0^\infty \frac{x^2 \cos 5x}{x^4 + 81}\, dx.$$

   Justify your steps.

4. (a) Let $a_0, a_1, \cdots, a_n$ be fixed complex numbers. Consider the two polynomials

$$p(z) = \sum_{k=0}^{n} a_k z^k = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0, \quad \text{and}$$

$$q(z) = \sum_{k=0}^{n} \overline{a_{n-k}} z^k = \overline{a_0} z^n + \overline{a_1} z^{n-1} + \cdots + \overline{a_n}$$

   in the complex plane. Show that, if $|a_n| < |a_0|$, then $p(z)$ and $\overline{a_0} p(z) - a_n q(z)$ have the same number of zeros in the unit ball $|z| < 1$. [Hint: First show that $|p(z)| = |q(z)|$ whenever $|z| = 1$.]

   (b) What is the number of zeros of the polynomial $z^4 + z^3 + z^2 + 2z + 2$ in the unit ball $|z| < 1$?

5. Let $\mathbf{D}$ denote the standard unit open ball in the complex plane $\mathbf{C}$, i.e., $\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$. Let $f : \mathbf{D} \to \mathbf{C}$ be a bounded analytic function such that $f(0) \neq 0$. Suppose $\{a_k\}_{k=1}^{\infty} \subset \mathbf{D}$ is a sequence of complex numbers in $\mathbf{D}$ such that $a_k \neq a_{k'}$ whenever $k \neq k'$, and $f(a_k) = 0$ for each k. Show that the series of real numbers

$$\sum_{k=1}^{\infty} \ln |a_k|$$

converges. (You may use freely without proof the following fact: For each $|a| < 1$, the linear fractional transformation f_a given by $f_a(z) = \dfrac{z - a}{1 - \bar{a}z}$ is analytic on $\mathbf{D}$, and $f_a(\mathbf{D}) = \mathbf{D}$.)

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2004-2005

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.