Ph.D. Qualifying Examination

Analysis

1. (a) Discuss the convergence, both pointwise and uniform, of

$$S_n(x) = \frac{nx}{1+n^2x^2}, \quad n = 1, 2, \dots .$$

on

(i) $[0, 1]$; and

(ii) $[c, 1]$, where $c > 0$.

(b) Let $S_{m,n} : [a, b] \to \mathbb{R}$, $m = 1, 2, \dots, n = 1, 2, \dots$. Suppose that

(i) for each n, $|S_{m,n}(x)| \le g_n(x)$ for all m and all $x \in [a, b]$;

(ii) $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on $[a, b]$; and

(iii) for each n, $S_{m,n}(x) \to S_n(x)$ on $[a, b]$ as $m \to \infty$.

Prove that $\sum_{n=1}^{\infty} S_n(x)$ converges uniformly on $[a, b]$.

2. (a) Let (X, ρ) be a metric space. Prove that (X, ρ) is compact if and only if every class of closed sets with finite intersection property has nonempty intersection. (A class of subsets of X is said to have the finite intersection property if every finite subclass has nonempty intersection.)

(b) Let $(\mathbb{R}^n, d_2)$ be the n-dimensional Euclidean space with the usual metric d_2 and $E = \prod_{i=1}^{n} [a_i, b_i]$ a compact subinterval in $\mathbb{R}^n$. Suppose δ is a positive function defined on E. Prove that there exists a finite collection $\{(I_i, x^{(i)})\}_{i=1}^m$ of interval-point pairs, where $x^{(i)} \in I_i \subseteq E$ for all i, such that for each i,

$$x^{(i)} \in I_i \subseteq B(x^{(i)}, \delta(x^{(i)})),$$

where $B(x^{(i)}, \delta(x^{(i)})) = \{y \in \mathbb{R}^n : d_2(x^{(i)}, y) < \delta(x^{(i)})\}$.
Hint: Proof by contradiction.

3. (a) Let (X, ρ) be a metric space with $x_0 \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = \rho(x, x_0)$. Prove that f is uniformly continuous on X.

   (b) Let (X, ρ) be a metric space and A a nonempty subset of X. Let $f(x) = dist(x, A) = \inf\{\rho(x, y) : y \in A\}$.

   Prove that $f : X \to \mathbb{R}$ is continuous. Is f uniformly continuous on X? Justify your answer.

   (c) Prove that in a separate metric space every uncountable set contains a convergent sequence of distinct points.

4. (a) A collection of continuous real-valued functions on a set $S \subseteq \mathbb{R}$ is said to be equicontinuous if for each $\varepsilon > 0$, there is a $\delta > 0$ so that $|f(x) - f(y)| \leq \varepsilon$ when f is in the collection, x, y are in S and $|x - y| \leq \delta$.

   (i) Is the collection $\{\cos nx, n = 1, 2, ...\}$ equicontinuous on $(-\infty, \infty)$? Justify your answer.

   (ii) Let $\{f_n\}$ be equicontinuous on $[a, b]$ and $f_n \to f$ uniformly on $[a, b] \cap \mathbb{Q}$. Prove that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$, where $\|g\|_\infty = \sup\{|g(x)| : x \in [a, b]\}$.

   (b) Prove that every compact metric space is separable.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2004-2005

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.
2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.