Ph.D. Qualifying Examination
2003/2004 Semester II
Linear Algebra

1. (a) A is an $n \times n$ matrix such that $A^2 - A - 2I = 0$. Show that $A + 2I$ is non-singular and find its inverse.

   (b) A is an $n \times n$ matrix such that $A = I - vv^T$ where v is a non-zero column vector in $\mathbb{R}^n$. Show that $A^2 = A$ if and only if $v^T v = 1$.

2. Let A be the $n \times n$ real matrix given by

$$A = \begin{pmatrix} 1 & a & a & \cdots & a \\ a & 1 & a & \cdots & a \\ a & a & 1 & \cdots & a \\ \vdots & & & \ddots & \vdots \\ a & a & \cdots & a & 1 \end{pmatrix}.$$

   What are the possible ranks of A? Justify your answer.

3. Let A be an $m \times n$ matrix with rank $A = r$. Suppose rank $AB = 1$ and none of the columns of AB are zero. Show that rank $B \leq n - r + 1$.

4. Let A be an $n \times n$ matrix with n distinct eigenvalues.

   (a) Suppose B is an $n \times n$ matrix with the same set of eigenvalues as A. Show that $A = QR$ and $B = RQ$ for some invertible matrix Q and matrix R.

   (b) Suppose every eigenvector of A is an eigenvector of C. Show that $AC = CA$.

5. Let A be an $n \times n$ real symmetric matrix. A is *positive definite* if $x^T Ax > 0$ for all non-zero column vectors $x \in \mathbb{R}^n$.

   (a) Show that A is positive definite if and only if $C^T AC = I$ for some invertible matrix C.

   (b) Show that, if A is positive definite, then $\det(A + I) > 1$.

6. Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional inner product space and T a linear operator on V. Show that there exists a unique linear operator $S : V \to V$ such that $\langle T(x), y\rangle = \langle x, S(y)\rangle$ for all $x, y \in V$.

END OF PAPER