Ph.D. Qualifying Examination
Complex Analysis

1. State the Cauchy Residue Theorem and use it to evaluate the integral

$$\int_0^\infty \frac{x^2 dx}{(x^2+1)(x^2+4)}.$$

2. Show that if $f(z)$ is an entire function and $\mathrm{Im} f(z) > \mathrm{Re} f(z)$ for all $z \in \mathbb{C}$, then f is a constant function.

3. State, without proof, Rouche's theorem. Use it to find the number of roots of the equation $e^{-z} + z^2 - 16 = 0$ in the right half plane Re $z > 0$. Justify your answer carefully.

4. (a) Use the power series representation of an analytic function to show that the zeroes of a non-constant analytic function are isolated.

   (b) Let $f(z)$ be an entire function which is represented by the series

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad (|z| < \infty).$$

   By differentiating the composite function $g(z) = f(f(z))$, find the first three non-zero terms for the Maclaurin series for $g(z)$ and deduce that the Maclaurin series for $\sin(\sin z)$ begins with

$$\sin(\sin z) = z - \frac{z^3}{3} + \cdots, \qquad (|z| < \infty).$$

5. Find a conformal map f which maps the infinite strip

$$S = \{z \in \mathbb{C} : -1 < \mathrm{Re} z < 1\}$$

to the unit disk

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

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