Ph.D. Qualifying Examination

Analysis

Notation

$(\mathbb{R}, d_1)$ denotes the metric space of real numbers with metric $d_1(x, y) = |x - y|$.

$(\mathbb{R}, d_p)$ denotes the n-dimensional Euclidean space with metric $d_p(x, y) = \left(\sum_{k=1}^{n} |x_k - y_k|^p\right)^{\frac{1}{p}}$, where $p \geq 1$.

(ℓ_p, d_p) denotes the metric space of real sequences $x = (x_k)_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} |x_k|^p < \infty$ and $d_p(x, y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^p\right)^{\frac{1}{p}}$, where $p \geq 1$.

$S_r(x)$ and $S_r[x]$ denote an open sphere and a closed sphere in a metric space resepctively.

$\overline{A}$ denotes the closure of A in a metric space.

1. (a) (i) Let $(X, \|\cdot\|)$ be a normed space. Prove that $\overline{S_r(x)} = S_r[x]$.

   (ii) Let $(X, \hat{d})$ be a discrete metric space. Does $\overline{S_r(x)} = S_r[x]$ hold? Justify your answer.

   (b) For each $n = 1, 2, ...,$ let $S_{\frac{1}{n}}(x_n)$ and $S_{\frac{1}{n}}[x_n]$ be an open sphere and a closed sphere respectively in a complete metric space (X, ρ). Suppose that for each n, $S_{\frac{1}{n+1}}(x_{n+1}) \subseteq S_{\frac{1}{n}}(x_n)$. Prove that $\bigcap_{n=1}^{\infty} S_{\frac{1}{n}}[x_n]$ is not empty.

2. (a) Let $f : ([0, 1], d_1) \to (\mathbb{R}, d_1)$. Suppose for each $x \in [0, 1]$, there exists $S_{r_x}(x) = \{y : |x - y| < r_x\}$ such that f is bounded on $S_{r_x}(x)$. Prove that f is bounded on $[0, 1]$.

   (b) (i) Let f and g be uniformly continuous on $A \subseteq \mathbb{R}$, under the standard distance d_1. Suppose that f and g are bounded on A. Prove that their product fg is uniformly continuous on A.

   (ii) Give an example to show that (i) does not hold if " f and g are bounded on A" is omitted.

3. (a) (i) Let A and B be subsets of $\mathbb{R}$. Suppose that A is compact and B is closed in $(\mathbb{R}, d_1)$. Prove that $A + B$ is closed, where $A + B = \{x + y : x \in A, y \in B\}$.

(ii) Give an example to show that (i) does not hold if "A is compact" is replaced by "A is closed".

(b) Let $(X, \hat{d})$ be a discrete metric space and $A \subseteq X$. Prove that if A is finite, then A is compact. Is the converse true? Justify your answer.

4. (a) Let $(x^{(n)})_{n=1}^{\infty}$ be a squence in $\mathbb{R}^m$. Prove that $d_p(x^{(n)}, x) \to 0$ as $n \to \infty$ if and only if $d_q(x^{(n)}, x) \to 0$ as $n \to \infty$, where p, $q \geq 1$.

(b) Let x, $y \in \mathbb{R}^n$ with $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ define

$$d_\infty(x, y) = \max\{|x_i - y_i| : i = 1, 2, ..., n\}.$$

Prove that

$$d_\infty(x, y) = \lim_{p \to \infty} d_p(x, y).$$

(c) Let $(x^{(n)})_{n=1}^{\infty}$ be a sequence in (ℓ_p, d_p) with $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, ...)$. Suppose that for each k, $\lim_{n \to \infty} x_k^{(n)} = x_k$ and there exists $y \in \ell_p$ such that $|x_k^{(n)}| \leq |y_k|$ for each k and n. Prove that $d_p(x^{(n)}, x) \to 0$ as $n \to \infty$.

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 2 2003-2004

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.