Ph.D. Qualifying Examination

Analysis

1. In this question, the metric d used is the usual metric $d(x, y) = |x - y|$.

   (i) Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$. Prove that f is uniformly continuous on D if and only if whenever $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in D with $d(x_n, y_n) \to 0$ as $n \to \infty$, we have $d(f(x_n), f(y_n)) \to 0$ as $n \to \infty$;

   (ii) Let $f : [0,1) \to \mathbb{R}$ be continuous. Is f uniformly continuous on $[0,1)$? Justify your answer; and

   (iii) Let $f : E \to \mathbb{R}$ be uniformly continuous. Is E closed and bounded? Justify your answer.

2. (a) Give four different kinds of metric defined on $\mathbb{R}^n$. (You do not have to justify your answer.);

   (b) Give a metric d defined on $\mathbb{R}^n$ such that $\|\alpha x\| \neq |\alpha| \, \|x\|$, where $\|y\| = d(y, 0)$; and

   (c) Let $S \subseteq \mathbb{R}$. Then S is said to have the Bolzano-Weierstrass property if every sequence in S has a convergent subsequence <u>with limit in S</u>.

      (i) Prove that, under the usual metric $d(x, y) = |x - y|$, S has the Bolzano-Weierstrass property if and only if S is bounded and closed.
      (You may use the fact that every bounded sequence has a convergent subsequence.)

      (ii) Is (i) true for any metric defined on $\mathbb{R}$? Justify your answer.

3. (i) Let $\alpha, \beta \geq 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that $\frac{\alpha^p}{p} + \frac{\beta^q}{q} \geq \alpha\beta$.

   (ii) Use (i) to prove the following Hölder inquality:

$$\sum_{i=1}^{n} |x_i y_i| \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q \right)^{\frac{1}{q}}.$$

4. Let $B[0,1]$ be the space of all bounded functions defined on $[0,1]$. On $B[0,1]$, define a metric d_∞ as follows:

$$d_\infty(f,g) = \sup\{|f(x) - g(x)| : x \in [0,1]\}.$$

Let $f_k \in B[0,1]$, $M_k \in \mathbb{R}$, $k = 1, 2, \ldots$, and $|f_k(x)| \le M_k$ for all $x \in [0,1]$ and all k. Suppose that $\sum_{k=1}^{\infty} M_k < \infty$. Prove that

(i) $\left(\sum_{k=1}^{n} f_k(x)\right)_{n=1}^{\infty}$ converges in $(B[0,1], d_\infty)$.

(ii) if each f_n is continuous on $[0,1]$, then $\sum_{k=1}^{\infty} f_k(x)$ is continuous on $[0,1]$; and

(iii) if each f_n is Riemann integrable on $[0,1]$, then $\sum_{k=1}^{\infty} f_k(x)$ is Riemann integrable on $[0,1]$ and $\sum_{k=1}^{\infty} \int_0^1 f_k(x)dx = \int_0^1 \sum_{k=1}^{\infty} f_k(x)dx.$

END OF PAPER

NATIONAL UNIVERSITY OF SINGAPORE

DEPARTMENT OF MATHEMATICS

SEMESTER 1 2003-2004

Ph.D. QUALIFYING EXAMINATION

PAPER 2

Time allowed : 3 hours

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions from **BOTH** sections.

2. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.